Finding Interpolating Curves Minimizing $L^\infty$ Acceleration in the Euclidean Space Via Optimal Control Theory

C. Yalçın Kaya*  J. Lyle Noakes†

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Abstract

We study the problem of finding an interpolating curve passing through prescribed points in the Euclidean space. The interpolating curve minimizes the pointwise maximum length, i.e., $L^\infty$-norm, of its acceleration. We re-formulate the problem as an optimal control problem and employ simple but effective tools of optimal control theory. We characterize solutions associated with singular and nonsingular controls. Some of the results we obtain are new even for the scalar interpolating function case. We reduce the infinite dimensional interpolation problem to an ensuing finite dimensional one and derive closed form expressions for interpolating curves. Consequently we devise efficient numerical techniques and illustrate them on examples.

Key words: Interpolation, splines, approximation, optimal control, singular control, discretization.

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1 Introduction

We consider the problem of finding an interpolating curve $z : [t_0, t_N] \rightarrow \mathbb{R}^n$, passing through $N+1$ points $p_0, p_1, \ldots, p_N$ in $\mathbb{R}^n$, specified at $t_0 < t_1 < \ldots < t_N$, with $N \geq 2$. The interpolating curve is required to minimize the maximum length of its second derivative, i.e., its acceleration, at every $t \in [t_0, t_N]$. Namely we consider the problem

\begin{equation}
\min \max_{t_0 \leq t \leq t_N} \| \ddot{z}(t) \| \quad \text{subject to} \quad z(t_0) = p_0, \ z(t_1) = p_1, \ldots, \ z(t_N) = p_N,
\end{equation}

where $\dddot{z}$ stands for the second derivative of $z$ and $\| \cdot \|$ the Euclidean norm. In the sequel, we refer to $\max_{t_0 \leq t \leq t_N} \| \dddot{z}(t) \|$ as the $L^\infty$-norm of the acceleration of $z$, or simply, the $L^\infty$ acceleration of $z$.

Interest in interpolating curves minimizing $L^\infty$ acceleration goes back to Favard’s early work in 1940 [14], albeit only for points $p_i$ prescribed in $\mathbb{R}$. The problem was studied further, still for points specified in $\mathbb{R}$, in the 1960s and 1970s – see, for example, de Boor [7, 8], Fisher and

*School of Mathematics and Statistics, University of South Australia, Mawson Lakes, S.A. 5095, Australia. E-mail: yalcin.kaya@unisa.edu.au.
†School of Mathematics and Statistics, University of Western Australia, WA 6009, Australia. E-mail: lyle@maths.uwa.edu.au.
Jerome [15], Karlin [22], and McClure [26]. While these studies consider the more general problem of minimizing \( \max_{0 \leq t \leq N} \| L z(t) \| \) where \( L \) is a linear differential operator (in Problem (P1) \( L \) is the 2nd derivative), they are mainly concerned with showing existence of a perfect spline solution. In these studies, a solution \( z \) is said to be a \textit{spline function of degree 2} if

(i) \( z(t) \) is quadratic in \( t \), i.e., \( \ddot{z}(t) \) is constant, in each interval \((t_{i-1}, t_i)\) and

(ii) \( z \in C^1[t_0, t_N] \).

A solution \( z \) is said to be a \textit{perfect spline}, if, in addition to (i) and (ii) above,

(iii) \( |\dddot{z}(t)| \) is constant for \( t \neq t_0, t_1, \ldots, t_N \).

Micchelli studied Problem (P1) for \( n \geq 1 \) in [28] in 1992, where he presented a result for minimizing the pointwise maximum length of the \( m \)th derivative of \( z(t) \), namely \( \max_{0 \leq t \leq t_N} \| z^{(m)}(t) \| \). We cite his result below for our setting, i.e., for \( m = 2 \).

Let \( \langle \cdot, \cdot \rangle \) denote the Euclidean inner product.

**Theorem 1** (Micchelli [28, Theorem 4.1]) \textit{There exists a curve \( z : [0,1] \to \mathbb{R}^n \) which solves Problem (P1) such that}

(i) \( \|\dddot{z}(t)\| \) is constant for a.e. \( t \in [t_0, t_N] \),

(ii) For every \( a \in \mathbb{R}^n \), the function \( \langle a, \dddot{z} \rangle \) has at most \( N - 2 \) sign changes.

For \( n = 1 \), Theorem 1 was proved by Karlin [22] in 1975, where the solution curve constitutes a perfect spline.

To the authors’ knowledge, not all possible solutions of Problem (P1) have yet been fully explored in the literature, even in the case when \( n = 1 \). For example, it is well-known to researchers in the area that, in some situations, there are solutions to Problem (P1) for which \( \|\dddot{z}(t)\| \) is not necessarily constant for a.e. \( t \in [t_0, t_N] \). We aim to explore these instances in more detail. Moreover, efficient numerical methods have not yet been developed even for the existing results pertaining to solutions with a constant length of the acceleration vector in \( \mathbb{R}^n \). We do not refer to the latter type of solutions in \( \mathbb{R}^n \) for \( n \geq 2 \) as perfect splines anymore, because, as it turns out, the interpolating curves in this case are not necessarily piecewise polynomial, let alone piecewise quadratic polynomial. Finding these nonpolynomial interpolating curves is clearly far more challenging analytically and computationally than polynomial ones.

In this paper, in order to explore all solutions of Problem (P1), we first transform Problem (P1) to an optimal control problem and then employ simple but effective tools of optimal control theory in order to understand the qualitative nature of the solutions as well as derive closed form expressions for the solutions. As a tool, we use, in particular, a maximum principle for multi-process control problems as described by Clarke and Vinter in [6] and in a more practical fashion by Augustin and Maurer in [3].

The idea of using optimal control theory for interpolation problems is not new. McClure [26] formulates an optimal control problem to answer the question of existence of a perfect spline. In his formulation of the problem, the question reduces to existence of bang–bang controls. We would also like to point to Aronsson’s study in [2] which extends McClure’s optimal control setting and question to problems where the objective to minimize is a general nonlinear functional of the derivatives of \( z(t) \), rather than \( \max_{0 \leq t \leq t_N} \| L z(t) \| \).
McClure’s formulation of an optimal control problem for interpolation differs from ours in two main aspects: (i) the interpolating function in question is only scalar, (ii) the formulation is not suitable for characterization of all possible solutions. As opposed to (i), our optimal control formulation is concerned with interpolating curves in $\mathbb{R}^n$ for $n \geq 1$ (including the scalar case). As opposed to (ii), we formulate the optimal control problem in a more direct fashion and incorporate the concept of multiprocess control so that we can write down the optimality conditions which in turn help us characterize solutions as well as furnish efficient numerical methods.

Micchelli, Smith, Swetits and Ward pave the way for tackling scalar interpolation problems with constraints in [27], where they require convexity of the interpolants. It is well-known that an interpolating curve minimizing the $L^2$-norm of its acceleration is a piecewise cubic spline. Dontchev, H. D. Qi, L. Qi, Kolmanovski and Yin [9, 12, 13] investigate such problems with various types of constraints such as “strips” between consecutive data points and convexity. Fredenhagen, Oberle and Opfer [16, 34] study restricted as well as monotone cubic spline interpolants by treating the interpolation problem as an optimal control problem. Agwu and Martin’s study in [1] is along similar lines. Interpolating curves minimizing various other criteria (but not the criterion we consider in this paper) are studied by means of optimal control by Isaev in [20].

Although we investigate an interpolation problem in the Euclidean space, it is worth mentioning that interpolation on differentiable manifolds (or non-Euclidean spaces) is an important class of problems, not only in theoretical, but also practical terms. Analogues of quadratic and cubic interpolants on manifolds have been intensively studied by researchers in the area, see e.g. [5, 29, 30, 31, 32, 33, 35]. Existing numerical techniques for finding interpolating curves on manifolds usually involve solving subproblems of finding interpolating curves in the Euclidean space via coordinate parameterization [19, 36, 21]. So interpolation in the Euclidean and non-Euclidean spaces are intrinsically related, and the results of our study in the Euclidean space might have an implication on interpolation on manifolds.

The way we formulate the optimal control problem (P3), or (P4), in the next section, for interpolation is reminiscent of that in [34]. However, because we minimize the $L^\infty$ acceleration of the interpolating curve, rather than the $L^2$ acceleration, the resulting optimal control problem has a path constraint on the control variables, even though the original problem has no such constraints, other than the point constraints given at the interpolation nodes. Since the Hamiltonian function for our problem is linear in the control variable, singular control turns out to be entirely possible.

In Theorem 2, we state, amongst other facts, that if the interpolating curve minimizing its $L^\infty$ acceleration is nonsingular, then the length of its acceleration is pointwise constant, say, a constant $\alpha$. In $\mathbb{R}$ (the scalar case), a nonsingular interpolating curve is obviously a perfect spline. We also state that if the interpolating curve is singular in an interval, then the length of its acceleration in that interval is pointwise less than or equal to $\alpha$. Therefore, in principle, there are infinitely many solutions if an interpolating curve is singular.

For the case of $\mathbb{R}^n$, $n \geq 2$, we provide in Theorem 3 a closed form expression for a nonsingular interpolating curve in terms of (elementary functions of) constant adjoint variable vectors and $\alpha$ which are found by solving a finite-dimensional optimization problem. In other words, Problem (P1), markedly more difficult than the case when $n = 1$, is reduced to the finite-dimensional optimization problem (Pfd). We illustrate in turn that Problem (Pfd) can be solved much more easily and accurately by standard optimization software.

In Theorem 4, we give conditions under which an interpolating curve is singular, or nonsingular, for the special case of $N = 3$ (four points). (Note that with three points, an interpolating
curve is always nonsingular and quadratic.) These conditions are straightforward to check, and they should lay the ground for singular curves for \( N \geq 4 \) (five of more points), which we leave for future work. Theorem 4 and its proof furnish a useful and easy-to-implement numerical method. Consequently we provide Algorithm 1, which constructs an interpolating curve, singular or nonsingular, for \( N = 3 \), efficiently. It should be noted that Theorem 4 and Theorem 2(e) are new results even for \( n = 1 \).

We present examples and numerical experiments to illustrate the working as well as efficiency of the new methods we propose. We write down exact solutions for singular interpolating curves in the case of four points. For \( N \geq 4 \), where we cannot employ Algorithm 1 anymore, we invoke Theorem 3 and the ensuing Problem (Pfd) to construct interpolating curves with the assumption that the interpolating curve is nonsingular and nonquadratic. Note that if one can find a solution to Problem (Pfd), it means that the interpolating curve found is indeed nonsingular and nonquadratic, as otherwise, if the underlying interpolating curve were singular or quadratic, there would have been no solution to Problem (Pfd).

The paper is organized as follows. In Section 2, we formulate the interpolation problem as an optimal control problem and obtain optimality conditions using a maximum principle for multiprocess control problems. In Section 3, we present results for nonsingular and singular interpolating curves. We also propose Algorithm 1 for interpolation with four points. In Section 4, we carry out numerical experiments using examples with singular and nonsingular interpolating curves passing through four or more points in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), and make comparisons with some direct discretization schemes to illustrate the efficiency of our techniques.

## 2 Formulation as an Optimal Control Problem

Using a standard trick from nonlinear programming, Problem (P1) can equivalently be written as

\[
\text{(P2)} \begin{cases} 
\min \alpha \\
\text{subject to } \|\ddot{z}(t)\| \leq \alpha, \quad \text{for all } t \in [t_0, t_N], \\
\quad z(t_0) = p_0, \quad z(t_1) = p_1, \ldots, \quad z(t_N) = p_N,
\end{cases}
\]

where \( \alpha \geq 0 \) is a new (constant) unknown parameter. As it stands, Problem (P2) is a nonsmooth, i.e., nondifferentiable, problem. The constraint \( \|\ddot{z}(t)\| \leq \alpha \) can be re-written as \( \|\ddot{z}(t)\|^2 \leq \alpha^2 \) to transform Problem (P2) to a smooth, i.e., differentiable, one; however we refrain from doing this and proceed with formulating Problem (P2) as an optimal control problem.

Let \( x_1(t) := z(t) \in \mathbb{R}^n \), \( x_2 := dz/dt(t) =: \dot{z}(t) \in \mathbb{R}^n \), and \( \alpha u(t) := \ddot{z}(t) \in \mathbb{R}^n \). Now Problem (P2) can be re-written as a parametric optimal control problem, with the parameter \( \alpha \) :

\[
\text{(P3)} \begin{cases} 
\min \alpha \\
\text{subject to } \dot{x}_1(t) = x_2(t), \quad x_1(t_0) = p_0, \quad x_1(t_1) = p_1, \ldots, \quad x_1(t_N) = p_N, \\
\dot{x}_2(t) = \alpha u(t), \quad \|u(t)\| \leq 1, \quad \text{for all } t \in [t_0, t_N].
\end{cases}
\]

Numerically speaking, Problem (P3) is in a convenient form for discretizing and then applying a direct (finite dimensional) optimization method to obtain an approximate solution, as we will indeed do in Section 4. Problem (P3) can be re-written in a “more standard” form, after defining
a new state variable, \( x_3(t) := \alpha \), as follows.

\[
(P4) \begin{cases}
\min_{t_0} \int_{t_0}^{t_N} x_3(t) \, dt \\
\text{subject to } \dot{x}_1(t) = x_2(t), \quad x_1(t_0) = p_0, \quad x_1(t_1) = p_1, \ldots, \quad x_1(t_N) = p_N, \\
\dot{x}_2(t) = x_3(t) u(t), \quad \|u(t)\| \leq 1, \quad \text{for all } t \in [t_0, t_N], \\
\dot{x}_3(t) = 0.
\end{cases}
\]

Problem (P4) can be transformed further into an optimal multiprocess (or multistage) control problem, for which a maximum principle (i.e., necessary conditions of optimality) is given by Clarke and Vinter [6]. Clarke and Vinter provide their maximum principle for very general systems, including systems which are not differentiable, for which the transversality conditions are presented by means of generalized derivatives and normal cones. Augustin and Maurer [3] transforms the multistage control problem for a special class of systems (including the class of system we have in this paper) into a single-stage one by means of a standard rescaling of the stage time durations (defined below). This allows the transversality conditions to be described rather more simply. We will make use of both references [3] and [6] in writing down the necessary conditions of optimality.

Define a new time variable \( s \) in terms of \( t \) as follows.

\[
t = t_{i-1} + s \tau_i, \quad s \in [0, 1], \quad \tau_i := t_i - t_{i-1}, \quad i = 1, \ldots, N.
\]

With this definition, the time horizon of each stage is re-scaled as \([0, 1]\) in the new time variable \( s \). Let

\[
x_j^i(s) := x_j(t) \quad \text{and} \quad u_j^i(s) := u(t) \quad \text{for} \quad s \in [0, 1], \quad t \in [t_{i-1}, t_i], \quad i = 1, \ldots, N, \quad j = 1, 2, 3.
\]

Here \( x_j^i \) and \( u_j^i \) denote the values of the state and control variables \( x_j \) and \( u \), respectively, in stage \( i \). In addition to the “interior” point constraints in (P4), with the usage of stages one needs to pose constraints to ensure continuity of the state variables at the junction of two stages; namely one needs to require

\[
x_j^i(1) = x_j^i+1(0), \quad i = 1, \ldots, N-1, \quad j = 1, 2, 3.
\]

### 2.1 The maximum principle

In this section, we state a maximum principle, i.e., necessary conditions of optimality, for Problem (P4), using [6, Theorem 3.1 and Corollary 3.1] and [3, Section 4]. Define the Hamiltonian function in the \( i \)th stage as

\[
H_j^i(x_1^i, x_2^i, x_3^i, u^i, \lambda_0^i, \lambda_1^i, \lambda_2^i, \lambda_3^i) := \lambda_0^i x_3^i + \langle \lambda_1^i, x_2^i \rangle + x_3^i \langle \lambda_2^i, u^i \rangle,
\]

where \( \lambda_0 \) is a scalar (multiplier) parameter, and \( \lambda_j^i : [0, 1] \to \mathbb{R}^n, \ j = 1, 2, 3 \), are the adjoint variable (multiplier) vectors in the \( i \)th stage. Let

\[
H_j^i[s] := H_j^i(x_1^i(s), x_2^i(s), x_3^i(s), u^i(s), \lambda_0^i, \lambda_1^i, \lambda_2^i, \lambda_3^i(s)).
\]

Suppose that \( x_i \in W^{1,\infty}(0, 1; \mathbb{R}^n), \ i = 1, 2, 3, \ u \in L^\infty(0, 1; \mathbb{R}^n), \) are optimal for Problem (P4). Then there exist a number \( \lambda_0 \geq 0 \), vector functions \( \lambda_j^i \in W^{1,\infty}(0, 1; \mathbb{R}^n), \ j = 1, 2, 3 \), and numbers \( \pi_i^j, i = 1, \ldots, N+2 \), such that \( \lambda_k^i(t) = (\lambda_0, \lambda_1^i(s), \lambda_2^i(s), \lambda_3^i(s)) \neq 0, \ i = 1, \ldots, N+2, \)
for every $s \in [0,1]$, and the following conditions hold for a.e. $s \in [0,1]$, in addition to the constraints given in Problem (P4).

\[
\dot{\lambda}_j(s) = -H_i^{[j]}[s], \quad \text{a.e. } s \in [0,1], \quad j = 1, 2, 3, \quad i = 1, \ldots, N, \quad (1)
\]

\[
\lambda_1^{[i+1]}(0) = \lambda_1^{[i]}(1) + \pi^{[i]}, \quad i = 2, \ldots, N - 1, \quad (2)
\]

\[
\lambda_j^{[0]}(0) = 0, \quad \lambda_j^{[1]}(1) = 0, \quad j = 2, 3, \quad i = 1, N, \quad (3)
\]

\[
\lambda_j^{[1]}(1) = \lambda_j^{[i+1]}(0), \quad j = 2, 3, \quad i = 1, \ldots, N - 1, \quad (4)
\]

\[
u_i^{[j]}(s) \in \arg\min_{\|v\| \leq 1} H_i^{[j]}(x_1^{[1]}(s), x_2^{[1]}(s), x_3^{[1]}(s), v, \lambda_0, \lambda_1^{[1]}(s), \lambda_2^{[1]}(s), \lambda_3^{[1]}(s)) \quad \text{a.e. } s \in [0,1], \quad (5)
\]

\[
H_i^{[j]}[s] = h_i^{[j]}, \quad \text{for all } s \in [0,1]. \quad (6)
\]

where $h_i^{[j]}, i = 1, \ldots, N$, are constants. Conditions (1)-(2) state that the values of the components of the adjoint variable vector $\lambda_1^{[i]}$ are constant but may have jumps going from one stage to the other. Condition (4), on the other hand, asserts continuity of the adjoint variables $\lambda_2^{[i]}$ and $\lambda_3^{[i]}$, going from one stage to a consecutive one. It should also be noted that the Hamiltonian function $H_i^{[j]}$ may have a different constant value in each stage.

For a neater appearance, we will re-write Conditions (1)-(6) and elaborate them further, by means of the overall state, control and adjoint variables. For this purpose, define the overall adjoint variables $\lambda_j(t), j = 1, 2, 3$, formed by concatenating the stage adjoint variables, as follows.

\[
\lambda_j(t) := \lambda_j^{[i]}(s), \quad t = t_{i-1} + s \tau_i, \quad s \in [0,1], \quad \tau_i := t_i - t_{i-1}, \quad i = 1, \ldots, N.
\]

Define the overall Hamiltonian function as

\[
H(x_1, x_2, x_3, u, \lambda_0, \lambda_1, \lambda_2, \lambda_3) := \lambda_0 x_3 + \lambda_1^T x_2 + x_3 \lambda_2^T u.
\]

Further, let

\[
H[t] := H(x_1(t), x_2(t), x_3(t), u(t), \lambda_0(t), \lambda_1(t), \lambda_2(t), \lambda_3(t)).
\]

Conditions (1)-(6), along with the state equations and constraints, can now be neatly re-written as follows.

\[
\dot{x}_1(t) = x_2(t), \quad x_1(t_0) = p_0, \quad x_1(t_1) = p_1, \ldots, \quad x_1(1) = p_N, \quad (7)
\]

\[
\dot{x}_2(t) = x_3(t) u(t), \quad (8)
\]

\[
\dot{x}_3(t) = 0 \quad \text{for all } t \in [t_{i-1}, t_i], \quad i = 1, \ldots, N + 2, \quad (9)
\]

\[
\lambda_1(t) = \lambda_1^{[i]}, \quad \text{for a.e. } t \in [0,1], \quad \lambda_2(0) = 0, \quad \lambda_2(1) = 0, \quad (10)
\]

\[
\dot{\lambda}_2(t) = -\lambda_1(t), \quad \text{for a.e. } t \in [0,1], \quad \lambda_2(0) = 0, \quad \lambda_2(1) = 0, \quad (11)
\]

\[
\dot{\lambda}_3(t) = -\lambda_0 - \lambda_2^T(t) u(t), \quad \text{for a.e. } t \in [0,1], \quad \lambda_3(0) = 0, \quad \lambda_3(1) = 0, \quad (12)
\]

\[
u(t) = \begin{cases} 
-\lambda_2(t)/\|\lambda_2(t)\|, & \text{if } \lambda_2(t) \neq 0, \\
\text{undetermined}, & \text{if } \lambda_2(t) = 0 \text{ for a.e. } t \in [s_1, s_2] \subset [0,1],
\end{cases}
\]

\[
H[t] = h_i^{[j]}, \quad \text{for all } t \in [t_{i-1}, t_i], \quad (13)
\]

where $\lambda_1^{[i]} \in \mathbb{R}^n$ and $h_i^{[j]} \in \mathbb{R}$, $i = 1, \ldots, N$, are constants. The control for the case when $\lambda_2(t) = 0$ for a.e. $t \in [s_1, s_2] \subset [t_0, t_N]$ is referred to as singular control; otherwise the control $u$ and state $x_1$ are said to be nonsingular. The solution curve $x_1$ is said to be totally singular if $\lambda_2(t) = 0$ for a.e. $t \in [t_0, t_N]$. Otherwise a singular $x_1$ is said to be partially singular.
3 Results

Let \( x_j^n : [t_{i-1}, t_i] \to \mathbb{R}^n \) be such that \( x_j^n(t) = x_j(t) \) for \( t \in [t_{i-1}, t_i] \), \( i = 1, \ldots, N \). Here \( x_j^n \) is referred to as \( x_j \) in stage \( i \). Let \( \lambda_j^i \) and \( u^i \) also be defined similarly. In the sequel, \( \lambda_2^i(t) \equiv 0 \) will mean that \( \lambda_2^i(t) = 0 \) for a.e. \( t \in [t_{i-1}, t_i] \). Suppose that \( x_1 \) solves (P4) and that \( \alpha \) is a resulting minimum \( L^\infty \)-norm of the acceleration and \( \lambda_2 \) an associated adjoint variable.

**Fact 1** \( \lambda_2^i(t) \neq 0 \) if, and only if, \( \lambda_2^i(t) = 0 \) at most for a single value of \( t \in [t_{i-1}, t_i] \).

Proof. The fact follows immediately from the linearity of \( \lambda_j^i \) in stage \( i \), by (10)-(11). \( \square \)

By Fact 1, the expression \( \lambda_2^i(t) \neq 0 \) can be considered equivalent in this context to “\( \lambda_2^i \) is almost nowhere zero in stage \( i \)”.

**Theorem 2** Let \( \alpha \) be the optimal value of Problem (P3). Then the following statements hold.

(a) The interpolating curve \( x_1 \) cannot be totally singular.

(b) If \( \lambda_2^i(t) \neq 0 \) then \( ||x_1^i(t)|| = \alpha \) for a.e. \( t \in [t_{i-1}, t_i] \).

(c) If \( \lambda_2^i(t) \neq 0 \) and \( \lambda_2^N(t) \neq 0 \) then \( x_1^i \) and \( x_1^N \) are quadratic.

(d) If \( ||x_1^i(t)|| < \alpha \) for a.e. \( t \in [s_1, s_2] \subset [t_{i-1}, t_i] \) then \( \lambda_2^i(t) = 0 \).

(e) If \( \lambda_2^i(t) \equiv 0 \) then \( ||x_1^i(t)|| \leq \alpha \) for a.e. \( t \in [t_{i-1}, t_i] \).

Proof.

(a) Suppose that \( \lambda_2^i(t) \equiv 0 \). Then, by (11) and (12), \( \lambda_1^i(t) \equiv 0 \) and \( \lambda_3^i(t) \equiv 0 \), i.e., \( \lambda(t) = (\lambda_1^i(t), \lambda_2^i(t), \lambda_3^i(t)) \equiv 0 \), contradicting the nontriviality of \( \lambda(t) \) by the maximum principle.

(b) Fix \( i \) such that \( \lambda_2^i(t) \neq 0 \). Then using (7), (8), (13) and Fact 1, one gets \( ||x_1^i(t)|| = ||x_2^i(t)|| = \alpha \) \( ||x_1^i(t)|| = ||x_2^i(t)|| \), which is a contradiction. Therefore \( x_2^i, i = 1, 2, \) are quadratic in \( t \).

(c) Note that \( x_3^i(t) = \alpha \), a constant. From (11) and (13),

\[
\begin{align*}
\lambda_1^i(t) &= -(t - t_0) \lambda_1^i; & u^i(t) = \lambda_1^i/||\lambda_1^i||; & \dot{x}_1^i(t) = \alpha u^i(t) = \alpha \lambda_1^i; \\
\lambda_2^N(t) &= (t_N - t_0) \lambda_1^N; & u^N(t) = -\lambda_1^N/||\lambda_1^N||; & \dot{x}_2^N(t) = \alpha u^N(t) = -\alpha \lambda_1^N;
\end{align*}
\]

where \( \lambda_1^1 \) and \( \lambda_1^N \) are the unit vectors along \( \lambda_1^i \) and \( \lambda_1^N \), respectively, which are constant. Therefore \( x_2^i, i = 1, 2, \) are linear in \( t \), and so (7) implies that \( x_1^i, i = 1, 2, \) are quadratic in \( t \).

(d) Suppose that \( ||x_1^i(t)|| < \alpha \) but \( \lambda_2^i(t) \neq 0 \) for a.e. \( t \in [s_1, s_2] \subset [t_{i-1}, t_i] \). Then \( ||x_1^i(t)|| = \alpha \) \( ||u(t)|| = \alpha \) \( ||x_2^i(t)|| = \alpha \), which is a contradiction. Therefore \( \lambda_2^i(t) = 0 \) for a.e. \( t \in [s_1, s_2] \subset [t_{i-1}, t_i] \) and so \( \lambda_1^i = 0 \), which together imply that \( \lambda_2^i(t) \equiv 0 \).

(e) By definition, \( \alpha = ||x_1||_{L^\infty} \). Therefore \( ||x_1^i(t)|| \leq \alpha \) for a.e. \( t \in [t_{i-1}, t_i] \).

\( \square \)

Theorem 2 states a number of useful facts by making use of optimal control theory. These facts can be re-iterated in more practical terms as follows: (a) The interpolating curve \( x_1 \), when singular, can only be partially singular; (b) if \( x_1 \) is nonsingular then the pointwise length of the
acceleration is constant throughout the curve; (c) in particular, if the first and last segments (or stages) of \( x_1 \) are nonsingular then these segments are nothing but curves quadratic in \( t \); (d) if a segment of \( x_1 \) has a pointwise acceleration length less than \( \alpha \), then that segment, or stage, is singular; (e) if a segment of \( x_1 \) is singular then the pointwise maximum length of the acceleration of that segment is less than or equal to \( \alpha \).

Recall that, by Theorem 1(i), there always exists a solution with \( \| \dot{x}_1(t) \| = \alpha \) for a.e. \( t \in [t_0, t_N] \), including the singular case. In most singular cases, however, a singular curve segment may have a pointwise maximum acceleration magnitude of, say, \( \bar{\alpha} \), which is less than \( \alpha \). In these cases, there would exist infinitely many singular interpolating curves with maximum pointwise acceleration magnitude ranging from \( \bar{\alpha} \) to \( \alpha \).

### 3.1 Interpolation With Three Points

An interpolating curve passing through three points, say, \( p_0, p_1, \) and \( p_2 \), in \( \mathbb{R} \), specified respectively at \( t_0, t_1, \) and \( t_2, \) has two stages, and so by Theorem 2(c) the curves in these stages are quadratic. In particular, \( \lambda_2(t) \neq 0 \). Before we write down the quadratic interpolating curves in closed form, we also express the adjoint variables, because the simple calculation sets the manner in which the adjoint variables are worked out for the cases with more than just three points specified. Using (10)-(11), one gets

\[
\lambda_1^1(t) = -(t - t_0) \lambda_1^1,
\]

and

\[
\lambda_2^2(t) = -\frac{(t_1 - t_0)(t_2 - t)}{(t_2 - t_1)} \lambda_1^1,
\]

where the adjoint variable \( \lambda_1^1 \) is a (nonzero) constant vector. Recall that \( \lambda_1^1 \) is defined for \( t \in [t_0, t_1] \), and \( \lambda_2^2 \) for \( t \in [t_1, t_2] \). In deriving the expressions above, we have used \( \lambda_1^1 = -[(t_1 - t_0)/(t_2 - t_1)] \lambda_1^1 \). By (13), optimal control in each stage is then given by

\[
u^1(t) = u^2(t) = \hat{\lambda}_1^1,
\]

where \( \hat{\lambda}_1^1 = \lambda_1^1/\| \lambda_1^1 \| \). The controls \( u^1 \) and \( u^2 \) imply that the accelerations in each stage are the same; namely that

\[
\dot{x}_1^2(t) = \alpha \lambda_1^1 = \dot{x}_2^2(t),
\]

in other words,

\[
x_1^2(t) = 2a t + b_1 \quad \text{and} \quad x_2^2(t) = 2a t + b_2,
\]

and further,

\[
x_1^1(t) = a t^2 + b_1 t + c_1 \quad \text{and} \quad x_2^1(t) = a t^2 + b_2 t + c_2,
\]

where \( a, b_1, b_2, c_1 \) and \( c_2 \) are constant vectors. Using the fact that \( x_1^1(t_0) = p_0, x_1^1(t_1) = p_1, x_1^1(t_2) = p_2, \) and \( x_1^1(t_1) = x_2^2(t_1) \), one gets \( b_1 = b_2 =: b \) and \( c_1 = c_2 =: c \). Then further manipulations yield the following proposition. In the proposition, we use the standard notation for forward divided differences

\[
[p_i, p_{i+1}] := \frac{1}{t_{i+1} - t_i} (p_{i+1} - p_i), \quad i = 0, 1; \quad [p_0, p_1, p_2] := \frac{1}{t_2 - t_0} ([p_1, p_2] - [p_0, p_1]).
\]

**Proposition 1** The interpolating curve passing through the three points, \( p_0, p_1, \) and \( p_2, \) in \( \mathbb{R}^n \), specified respectively at \( t_0, t_1, \) and \( t_2, \) such that \( t_0 < t_1 < t_2, \) is given by the quadratic

\[
z(t) = at^2 + bt + c,
\]

where

\[
a = [p_0, p_1, p_2], \quad b = [p_0, p_1] - (t_0 + t_1) [p_0, p_1, p_2], \quad c = p_0 - t_0 ([p_0, p_1] - t_1 [p_0, p_1, p_2]).
\]
Remark 1 For the special case when \( h := t_{i+1} - t_i, i = 0, 1, \) and \( t_0 = 0, \) the constants in Proposition 1 reduce to the simpler expressions

\[
\begin{align*}
a &= \frac{1}{2h^2} (p_2 - 2p_1 + p_0), \\
 b &= \frac{1}{2h} (p_2 - p_0), \\
c &= p_0.
\end{align*}
\]

3.2 Interpolation With More Than Three Points

Recall that, for the case of three specified points, the interpolating curve \( z(t) \) is nonsingular, i.e., \( \lambda_1^2(t) \neq 0 \) and \( \lambda_2^2(t) \neq 0, \) and the solution is simply given as in Proposition 1. In this section, we look at the case of more than three specified points.

In Section 3.2.1 below, we consider the case when an interpolating curve, with \( N + 1 \) specified points, \( N \geq 2, \) is nonsingular. In the subsequent Section 3.2.2, we look at the special case of singular interpolating curves with four specified points. In both sections, we develop expressions for the solutions of the interpolating curves. The results of Sections 3.2.1 and 3.2.2 fully characterize the interpolating curves for four specified points in \( \mathbb{R}^n. \) We employ these results and expressions in Algorithm 1 in Section 3.2.2 to construct singular as well as nonsingular interpolating curves.

Our discussion in Sections 3.2.1 and 3.2.2 can possibly be used as a basis for extensions to the case of singular interpolating curves for more than four specified points.

3.2.1 Nonsingular interpolating curves

In this section, we derive closed form expressions for nonsingular interpolating curves in terms of \( \alpha \) and \( \lambda_i^2, \) which satisfy the optimality conditions (7)-(13).

From Theorem 2(c) and its proof, we have

\[
\begin{align*}
\dot{x}_1^1(t) &= \alpha \frac{\lambda_1^1}{\|\lambda_1^1\|}, \\
\dot{x}_2^N(t) &= -\alpha \frac{\lambda_N^1}{\|\lambda_N^1\|}.
\end{align*}
\]  
(15)

The expressions in (15) and manipulations similar to those performed for Proposition 1 yield the following proposition.

Proposition 2 The nonsingular interpolating curve segments in stages 1 and \( N \) are given by the quadratics

\[
\begin{align*}
x_1^1(t) &= a_1 t^2 + b_1 t + c_1, \\
x_N^1(t) &= a_N t^2 + b_N t + c_N,
\end{align*}
\]  
(16)

where

\[
\begin{align*}
a_1 &= \frac{\alpha}{2} \frac{\lambda_1^1}{\|\lambda_1^1\|}, \\
 a_N &= -\frac{\alpha}{2} \frac{\lambda_N^1}{\|\lambda_N^1\|}, \\
b_i &= \frac{1}{t_i - t_{i-1}} (p_i - p_{i-1}) - a_i (t_i + t_{i-1}), \\
c_i &= p_i - a_i t_i^2 - b_i t_i,
\end{align*}
\]  
(17)  
(18)  
(19)

with \( i = 1, N. \)
Recall that, by using (11), \( \lambda_2^i(t) = -(t-t_0) \lambda_1^i \). Furthermore, by using (11) again,

\[
\lambda_2^i(t) = \lambda_2^i(t_1) - (t-t_0) \lambda_1^2 = -(t_1-t_0) \lambda_1^1 - (t-t_1) \lambda_1^1.
\]

Proceeding inductively, we get

\[
\lambda_2^i(t_{i-1}) = - \sum_{k=1}^{i-1} (t_k - t_{k-1}) \lambda_1^k,
\]  
(20)

and so

\[
\lambda_2^i(t) = - \sum_{k=1}^{i-1} (t_k - t_{k-1}) \lambda_1^k - (t-t_{i-1}) \lambda_1^i, \quad i = 1, \ldots, N.
\]  
(21)

Note that, because \( \lambda_2^N(t_N) = 0 \),

\[
\sum_{k=1}^{N} (t_k - t_{k-1}) \lambda_1^k = 0.
\]

Let

\[
w^i = \lambda_2^i(t_{i-1}) + t_{i-1} \lambda_1^i.
\]  
(22)

Then, also by (20), \( \lambda_2^i(t) = w^i - t \lambda_1^i \), and so

\[
u^i(t) = \frac{\lambda_2^i(t)}{\|\lambda_2^i(t)\|} = \frac{w^i - t \lambda_1^i}{\|w^i - t \lambda_1^i\|} = \frac{t \lambda_1^i - w^i}{\sqrt{\|w^i\|^2 - 2 \langle w^i, \lambda_1^i \rangle t + \|\lambda_1^i\|^2 t^2}}.
\]

Also let

\[
r^i := \frac{\langle w^i, \lambda_1^i \rangle}{\|\lambda_1^i\|^2}, \quad s^i := \frac{\|w^i\|^2}{\|\lambda_1^i\|^2}.
\]  
(23)

Now, for \( t \in [t_{i-1}, t_i] \), \( i = 2, \ldots, N-1 \),

\[
F(t) := \int u^i(t) \, dt = \frac{1}{\|\lambda_1^i\|} \left( \lambda_1^i \int \frac{t \, dt}{\sqrt{t^2 - 2 r^i t + s^i}} - w^i \int \frac{dt}{\sqrt{t^2 - 2 r^i t + s^i}} \right).
\]

These integrals can be solved in terms of elementary functions to get

\[
F(t) = \frac{1}{\|\lambda_1^i\|} \left[ \lambda_1^i \sqrt{t^2 - 2 r^i t + s^i} + (r^i \lambda_1^i - w^i) \ln \left( t - r^i + \sqrt{t^2 - 2 r^i t + s^i} \right) \right].
\]  
(24)

The first term in (24) is well-defined, because \( \lambda_2^i(t) \neq 0 \) by nonsingularity, and so \( \lambda_1^i \neq 0 \) and \( t^2 - 2 r^i t + s^i = \|\lambda_2^i(t)\|^2/\|\lambda_1^i\|^2 \geq 0 \) for all \( t \in [t_{i-1}, t_i] \). The second (logarithm) term is well-defined by Proposition 3 further below for nonquadratic nonsingular interpolating curves.

**Lemma 1** Suppose that \( \lambda_2^i(\tau) \) is a scalar multiple of \( \lambda_1^i \) for some \( \tau \in [t_{i-1}, t_i] \).

(a) \( \lambda_2^i(t) \) is a scalar multiple of \( \lambda_1^i \) for all \( t \in [t_{i-1}, t_i] \).

(b) If \( \lambda_1^i \neq 0 \), then \( x_1^i \) is quadratic in \( t \).

(c) If \( \lambda_2^i(\tau) = 0 \) for some \( \tau \in [t_{i-1}, t_i] \) and \( \lambda_1^i \neq 0 \), then \( \lambda_2^i(t) \neq 0 \) for all \( t \in [t_{i-1}, t_i] \setminus \{\tau\} \). In particular, the acceleration vector \( \ddot{x}_1^i(t) \) switches from \(-\alpha \dot{x}_1^i\) to \( \alpha \dot{x}_1^i \) at \( t = \tau \).
By part (a), \( \hat{\lambda}_2(\tau) = \beta_2^T \lambda_1 \), \( \beta_2^T \) a real number.

From \( \lambda_2^2(\tau) = \lambda_2^2(t) - (\tau - t) \lambda_1^2 \), one simply gets \( \lambda_2^2(t) = \beta_2^T \lambda_1^2 \), with \( \beta_2^T = (\beta_2^T + \tau - t) \).

This proves part (a).

By part (a), \( \bar{x}_1^2(t) = -\alpha \lambda_2^2(t)/\|\lambda_2^2(t)\| = -\text{sign}(\beta_2^T + \tau - t) \alpha \lambda_1^2/\|\lambda_1^2\| \) for all \( t \in [t_{i-1}, t_i] \), except at \( t = \beta_2^T + \tau \). This proves part (b).

Set \( \beta_2^T = 0 \) in the proof of part (a). Then \( \lambda_2^2(\tau) = 0 \) and \( \lambda_2^2(t) = (\tau - t) \lambda_1^2 \), which implies that \( \lambda_2^2(t) \neq 0 \) for all \( t \neq \tau \). Furthermore, if \( \tau > t_{i-1} \) we get \( \bar{x}_1^2(t) = -\alpha \lambda_2^2(t)/\|\lambda_2^2(t)\| = -\alpha \lambda_1^2/\|\lambda_1^2\| = -\alpha \hat{\lambda}_1^2 \) for all \( t \in [t_{i-1}, \tau) \) and if \( \tau < t_i \) we get \( \bar{x}_1^2(t) = \alpha \hat{\lambda}_1^2 \) for all \( t \in (\tau, t_i] \). This proves part (c). \( \square \)

Remark 2 Lemma 1(a) equivalently asserts that if \( \lambda_2^2(\tau) \) is not a scalar multiple of \( \lambda_1^2 \) for some \( \tau \in [t_{i-1}, t_i] \), then \( \bar{x}_1^2(t) \) is not a scalar multiple of \( \lambda_1^2 \) for any \( t \in [t_{i-1}, t_i] \). In this case, in particular, \( \lambda_2^2(t) \neq 0 \) for any \( t \in [t_{i-1}, t_i] \). On the other hand, we see from Lemma 1(c) that if \( \lambda_2^2(\tau) = 0 \) for some \( \tau \in [t_{i-1}, t_i] \), then \( t = \tau \) is the only point where \( \lambda_2^2(t) = 0 \). It is also interesting to note in Lemma 1(c) that the acceleration vector of the quadratic interpolating curve in a stage switches between a constant vector and its negative at most once; in particular, the control variable \( u(t) \) would switch between \(-\lambda_1^2 \) and \( \lambda_1^2 \), if at all. The type of the control \( u(t) \) here is referred to as \textit{bang-bang} control in the optimal control literature. In the scalar case (i.e., when \( p_i \in \mathbb{R} \)), a switching point is nothing but an inflection point for \( x_1^2 \).

Proposition 3

(a) If \( \lambda_1^2 \neq 0 \) and \( x_1^2 \) is not quadratic in \( t \), then \( t-r_1^2+\sqrt{t^2-2r_1^2 t+s} > 0 \) for all \( t \in [t_{i-1}, t_i] \);

(b) If \( t-r_1^2+\sqrt{t^2-2r_1^2 t+s} > 0 \), then \( \lambda_2^2(t) \neq \beta_2^T \lambda_1^2, \beta_2^T \) a positive real number dependent on \( t \).

Proof. Through direct substitution and manipulations one can show that

\[ t-r_1^2+\sqrt{t^2-2r_1^2 t+s} = \frac{1}{\|\lambda_1^2\|} \left( \|\lambda_2^2(t)\| \|\lambda_1^2\| - \langle \lambda_2^2(t), \lambda_1^2 \rangle \right). \]

By the Cauchy-Schwarz inequality,

\[ 0 \leq \|\lambda_2^2(t)\| \|\lambda_1^2\| - \langle \lambda_2^2(t), \lambda_1^2 \rangle \leq 2 \|\lambda_2^2(t)\| \|\lambda_1^2\|. \]  

Suppose that \( \lambda_1^2 \neq 0 \) and \( x_1^2 \) is not quadratic in \( t \). Then, by Lemma 1(b) and Remark 2, \( \lambda_2^2(t) \neq \beta_2^T \lambda_1^2, \beta_2^T \) a real number dependent on \( t \), for all \( t \in [t_{i-1}, t_i] \). Therefore the first inequality in (25) is strict for all \( t \in [t_{i-1}, t_i] \), proving part (a).

Suppose that \( t-r_1^2+\sqrt{t^2-2r_1^2 t+s} > 0 \); in other words, by the first inequality in (25), \( \langle \lambda_2^2(t), \lambda_1^2 \rangle < \|\lambda_2^2(t)\| \|\lambda_1^2\| \). Then \( \lambda_2^2(t) \neq \beta_2^T \lambda_1^2, \beta_2^T \) a positive real number dependent on \( t \), proving part (b). \( \square \)

Remark 3 Lemma 2(b) and Proposition 3(a) characterize nonsingular solutions in the \( i \)th stage:

(i) \textit{Quadratic nonsingular interpolating curve}: In this case, \( \lambda_2^2(\tau) \) is a scalar multiple of \( \lambda_1^2 \), and (24) is not needed (although (24) is well defined if \( \lambda_2^2(\tau) \) is a negative scalar multiple of \( \lambda_1^2 \)).

(ii) \textit{Nonquadratic nonsingular interpolating curve}: In this case, \( \lambda_2^2(\tau) \) is not a scalar multiple of \( \lambda_1^2 \), and so (24) is well-defined. Note that, now, \( \lambda_2^2(t) \) is never zero in the interval \([t_{i-1}, t_i] \), either.
When $p_i \in \mathbb{R}$ (the scalar case), one only has the situation (i) above for nonsingular solutions, i.e., interpolating curves are quadratic. However, when $p_i \in \mathbb{R}^n$, $n > 1$, nonsingular interpolating curves can be nonquadratic as well as quadratic.

The second integral of $u$ is found similarly as
\[
G(t) := \int F(t) \, dt = \frac{1}{\|x_1^i\|} \left\{ \left( \frac{1}{2} (t - 3 \, r^i) \lambda_1^i + w^i \right) \sqrt{t^2 - 2 \, r^i \, t + s^i} \right.
+ \left. \frac{1}{2} (s^i - 3 \, (r^i)^2 + 2 \, r^i \, t) \lambda_1^i - (t - r^i) \, w^i \right\} \ln \left( t - r^i + \sqrt{t^2 - 2 \, r^i \, t + s^i} \right) \right) .
\] (26)

Just as in the case of $F(t)$ in (24), $G(t)$ in (26), which has the same type of terms, is also well-defined for nonquadratic nonsingular interpolating curve.

**Theorem 3** Suppose that $x_1^i$ is nonquadratic nonsingular interpolating curve in $t$, $i = 2, \ldots, N - 1$. Then the acceleration, velocity and position curves in each stage $i$, in addition to those given in Proposition 2, are expressed in terms of $\lambda_1^i$ and $\alpha$ as
\[
\dot{x}_1^i(t) = \frac{\alpha}{\|x_1^i\|} \frac{t \lambda_1^i - w^i}{\sqrt{t^2 - 2 \, r^i \, t + s^i}},
\] (27)
\[
\ddot{x}_1^i(t) = \dot{x}_1^{i-1}(t_{i-1}) + \alpha [F(t) - F(t_{i-1})], \quad \text{with} \quad \dot{x}_1^{i}(t_1) = 2 \, a_1 \, t_1 + b_1 ,
\] (28)
\[
\ddot{x}_1^i(t) = p_i - 1 + \left[ \dot{x}_1^{i-1}(t_{i-1}) - \alpha F(t_{i-1}) \right] (t - t_{i-1}) + \alpha \left[ G(t) - G(t_{i-1}) \right] ,
\] (29)
for $i = 2, \ldots, N - 1$, where $w^i$, $r^i$, $s^i$, $F(t)$ and $G(t)$ are as defined in (22)-(24) and (26), all in terms of $\lambda_1^i$ and $\alpha$, and $a_1$ and $b_1$ are as given in (17)-(18).

For simplicity in appearance, let the velocities at the nodes (i.e., at the junctions of the stages) be denoted by $v_i := \dot{x}_1(t_i)$. Then $v_i$ can simply be expressed, in terms $\lambda_1^i$ and $\alpha$, as
\[
v_i = v_{i-1} + \alpha [F(t_i) - F(t_{i-1})] , \quad i = 1, \ldots, N - 1 , \quad \text{with} \quad v_1 = 2 \, a_1 \, t_1 + b_1 ,
\] (30)
where $a_1$, $b_1$ and $F(t)$ are as given in (17)-(18) and (24).

For the case of nonquadratic, nonsingular interpolating curves for each stage $i$, $i = 2, \ldots, N - 1$, the infinite dimensional optimization problem (P3), or indeed (P1), can now be reduced to a finite $(nN + 1)$ dimensional optimization problem of finding $\lambda_1^i$ and $\alpha$ as follows.

\[
\begin{align*}
\text{(Pfd)} \quad \min & \quad \alpha \\
\text{subject to} & \quad p_i = p_{i-1} + [v_{i-1} - \alpha F(t_{i-1})] \, (t_i - t_{i-1}) + \alpha \, [G(t_i) - G(t_{i-1})] , \\
& \quad i = 2, \ldots, N - 1 , \\
& \quad v_{N-1} = 2 \, a_N \, t_{N-1} + b_N , \\
& \quad \sum_{k=1}^{N} (t_k - t_{k-1}) \lambda_1^k = 0 , \quad \alpha \geq 0 ,
\end{align*}
\]

where $a_N$, $b_N$, $w^i$, $r^i$, $s^i$, $F(t)$, $G(t)$ and $v_{N-1}$ are given in (17)-(18), (22)-(26) and (30), all in terms of $\lambda_1^i$ and $\alpha$.

It is obvious that if an interpolating curve is singular, i.e., if $\lambda_1^i = 0$ for some $i$, $i = 1, \ldots, N$, then a solution of Problem (Pfd) cannot be found. If we can find a solution of Problem (Pfd), then necessarily $\lambda_1^i \neq 0$, and that we have found the required solution. So in a situation where one cannot be sure if the interpolating curve is singular or nonsingular, then one can simply attempt solving Problem (Pfd).
Proposition 4 If there exists a solution to Problem (Pfd), then the interpolating curve is non-singular and nonquadratic.

With randomly generated initial guesses for a solver in the process of obtaining numerical solutions of Problem (Pfd), one is likely to get a different solution each time. In order to eliminate this “surprise element,” we will normalize $\lambda_i^1$, i.e., we will divide all components of $\lambda_i^1$, $i = 1, \ldots, N$, by $\max_{1 \leq i \leq n} |\lambda_i^1|$, in which case the solution for $\lambda_i^1$ is found uniquely.

3.2.2 Singular interpolating curves with four points

Lemma 2 Suppose that $p_0$, $p_1$, $p_2$, and $p_3$ are specified respectively at $t_0$, $t_1$, $t_2$, and $t_3$. Then one of the instances below must eventuate.

(i) $\lambda_1^2(t) \neq 0$, $\lambda_2^2(t) \neq 0$, $\lambda_3^2(t) \neq 0$;

(ii) $\lambda_1^2(t) \neq 0$, $\lambda_2^2(t) \neq 0$, $\lambda_3^2(t) \equiv 0$;

(iii) $\lambda_1^2(t) \equiv 0$, $\lambda_2^2(t) \neq 0$, $\lambda_3^2(t) \neq 0$.

Proof. Note that $\lambda_3^2(t) \equiv 0$ yields a totally singular interpolating curve, which is not possible by Theorem 2(a). Therefore one must always have that $\lambda_3^2(t) \neq 0$. By the same reason, we also cannot have $\lambda_1^2(t) \equiv 0$ and $\lambda_3^2(t) \equiv 0$ concurrently. This leaves us only with the instances (i)-(iii).

In the instance (i), the interpolating curves in all stages are nonsingular. This situation has already been dealt with in Section 3.2.1 and will be considered in Step 3 of Algorithm 1. The instances (ii) and (iii) will be studied in Theorem 4, the results of which will then be implemented in Steps 1 and 2 of Algorithm 1.

The following lemma is needed in both the proof of the forthcoming Theorem 4 and in the construction of singular interpolating curves, as described in Algorithm 1.

Lemma 3 For $v \in \mathbb{R}^n$, consider the problem

$$
(P5) \begin{cases} 
\min_{t_0 \leq t \leq t_1} \max_{t_0 \leq t \leq t_1} \|\ddot{z}(t)\| \\
\text{subject to } z(t_0) = p_0, \ z(t_1) = p_1, \ \dot{z}(t_0) = v.
\end{cases}
$$

The curve that solves Problem (P5) is quadratic. In particular,

$$
z(t) = \ddot{a} (t - t_0)^2 + v (t - t_0) + p_0,
$$

where

$$
\ddot{a} = \frac{1}{(t_1 - t_0)^2} (p_1 - p_0) - \frac{1}{(t_1 - t_0)} v.
$$

Proof. In the same way Problem (P1) was equivalently rewritten as Problem (P4), we can rewrite Problem (P5) equivalently as

$$
(P6) \begin{cases} 
\min_{t_0} \int_{t_0}^{t_1} x_3(t) \ dt \\
\text{subject to } \dot{x}_1(t) = x_2(t), \ x_1(t_0) = p_0, \ x_1(t_1) = p_1,
\end{cases}
$$

$$
\dot{x}_2(t) = x_3(t) u(t), \ x_2(t_0) = v, \ \|u(t)\| \leq 1,
$$

$$
\dot{x}_3(t) = 0.
$$
With the Hamiltonian defined as

\[ H(x_1, x_2, x_3, u, \lambda_0, \lambda_1, \lambda_2, \lambda_3) = \lambda_0 x_3 + \langle \lambda_1, x_2 \rangle + \langle \lambda_2, u \rangle, \]

the necessary optimality conditions imply that \( \lambda_1 \) is constant, \( \lambda_2(t) = (t_1 - t) \lambda_1 \), and \( u(t) = -\hat{\lambda}_1 \), where \( \hat{\lambda}_1 = \lambda_1 / \| \lambda_1 \| \). Now, since \( x_3 \) is constant and so \( \dot{x}_2(t) = -x_3 \hat{\lambda}_1 \) is a constant vector, we conclude that \( x_1 \) is quadratic. Integrating \( \dot{x}_2(t) \) and substituting the boundary conditions, one obtains the required expression for \( z(t) = x_1(t) \).

The following definitions are needed for Theorem 4 and Algorithm 1 below.

\[
a_L = [p_0, p_1, p_2], \tag{31}
\]
\[
b_L = [p_0, p_1] - (t_0 + t_1) [p_0, p_1, p_2], \tag{32}
\]
\[
c_L = p_0 - t_0 ([p_0, p_1] - t_1 [p_0, p_1, p_2]), \tag{33}
\]
\[
v_L = 2 a_L t_2 + b_L, \tag{34}
\]
\[
p^c_2 = p_2 + (t_3 - t_2) v_L, \tag{35}
\]
\[
a_R = [p_3, p_2, p_1], \tag{36}
\]
\[
b_R = [p_3, p_2] - (t_2 + t_3) [p_3, p_2, p_1], \tag{37}
\]
\[
c_R = p_3 - t_3 ([p_3, p_2] - t_2 [p_3, p_2, p_1]), \tag{38}
\]
\[
v_R = 2 a_R t_1 + b_R, \tag{39}
\]
\[
p^c_0 = p_1 + (t_0 - t_1) v_R, \tag{40}
\]

**Theorem 4**

(a) If \( \lambda^2_2(t) \equiv 0 \) then \( \| p_3 - p^c_2 \| \leq (t_3 - t_2)^2 \| a_L \| ; \)

if \( \| p_3 - p^c_2 \| < (t_3 - t_2)^2 \| a_L \| \) then \( \lambda^2_2(t) \equiv 0 \).

(b) If \( \lambda^2_2(t) \equiv 0 \) then \( \| p_0 - p^c_0 \| \leq (t_1 - t_0)^2 \| a_R \| ; \)

if \( \| p_0 - p^c_0 \| < (t_1 - t_0)^2 \| a_R \| \) then \( \lambda^2_2(t) \equiv 0 \).

Proof. We only prove part (a). Part (b) can be proved similarly to part (a), using symmetry and similar arguments.

Suppose that \( \lambda^2_2(t) \equiv 0 \). Then, with the points \( p_0, p_1 \) and \( p_2 \) respectively given at \( t_0, t_1 \) and \( t_2 \), the boundary and interior conditions of the 3-point interpolation problem in Section 3.1 are satisfied, where \( \lambda(t_0) = \lambda(t_2) = 0 \). Therefore, by Proposition 1,

\[
x_1(t) = a_L t^2 + b_L t + c_L, \quad t \in [t_0, t_2], \tag{41}
\]

where the constant vectors \( a_L, b_L \) and \( c_L \) are defined in (31)-(33). Note that the curve \( x_1(t) \) has velocity \( v_L \) at \( t_2 \), defined in (34). Then, taking the track sum, any curve joining \( p_2 \) and \( p_3 \) with a maximum pointwise (i.e., \( L^\infty \)) acceleration less than or equal to \( 2 \| a_L \| \) and with the same velocity \( v_L \) at \( t_2 \) defines an interpolating curve. Of all these curves, the quadratic interpolating curve has the smallest \( L^\infty \) acceleration by Lemma 3 (in the lemma, replace \( p_0 \) and \( p_1 \) by \( p_2 \) and \( p_3 \), \( t_0 \) and \( t_1 \) by \( t_2 \) and \( t_3 \), and \( v \) by \( v_L \)). Namely, we have

\[
x^3_1(t) = \tilde{a} t^2 + \tilde{b} t + \tilde{c},
\]
where \( \tilde{a}, \tilde{b} \) and \( \tilde{c} \) are constant vectors. Therefore one has, by Theorem 2(a)-(b),

\[
\|\tilde{a}\| \leq \|a_L\|.
\]

(42)

Because the velocities at \( t_2 \) have to agree,

\[
\dot{x}_1^2(t_2) = 2\tilde{a}t_2 + \tilde{b} = v_L, \quad \text{i.e.,} \quad \tilde{b} = v_L - 2\tilde{a}t_2.
\]

Furthermore,

\[
\begin{align*}
x_1^3(t_2) &= \tilde{a}t_2^2 + \tilde{b}t_2 + \tilde{c} = p_2 \\
x_1^3(t_3) &= \tilde{a}t_3^2 + \tilde{b}t_3 + \tilde{c} = p_3
\end{align*}
\]

Subtracting side by side,

\[
\begin{align*}
p_3 - p_2 &= \frac{1}{t_3 - t_2} (p_3 - p_2) \\
&= \frac{1}{t_3 - t_2} (\tilde{a}(t_3^2 - t_2^2) + \tilde{b}(t_3 - t_2)) \\
&= \frac{1}{t_3 - t_2} (\tilde{a}(t_3 + t_2) + \tilde{b}) \\
&= \tilde{a}(t_3 + t_2) + v_L - 2\tilde{a}t_2 \\
&= \tilde{a}(t_3 - t_2) + v_L.
\end{align*}
\]

Finally, rearranging, one gets

\[
\tilde{a} = \frac{1}{(t_3 - t_2)^2} (p_3 - p_3^c),
\]

where \( p_3^c \) is defined in (35). Then, by (42), it follows that

\[
\|p_3 - p_3^c\| \leq (t_3 - t_2)^2 \|a_L\|.
\]

This establishes the first statement of part (a). A schematic illustration of the concepts that have been used in the proof so far is given in Figure 1.

The second statement of part (a) is proved as follows. Let

\[
\|p_3 - p_3^c\| < (t_3 - t_2)^2 \|a_L\|.
\]

(43)

Let Problem (P1a) denote Problem (P1) with the three points \( p_0, p_1 \) and \( p_2 \), given at \( t_0, t_1 \) and \( t_2 \). By Proposition 1, there exists a unique solution \( y_1 \) to Problem (P1a), which is a quadratic interpolating curve with the \( L^\infty \) acceleration \( \alpha_1 = 2\|a_L\|. \)

Now consider Problem (P1) with the four points \( p_0, p_1, p_2 \) and \( p_3 \), respectively at \( t_0, t_1, t_2 \) and \( t_3 \), (by incorporating \( p_3 \) at \( t_3 \) in Problem (P1a)) and denote it by Problem (P1b). Since
\( \alpha = 2 \| a_L \| \) for the solution of Problem (P1a), any solution of Problem (P1b) obviously has an \( L^\infty \) acceleration \( \alpha \geq 2 \| a_L \| \). It is possible to construct a curve \( x_1 \) passing through \( p_0, p_1, p_2 \) and \( p_3 \) as in the first part of the proof: take the quadratic solution \( y_1 \) of Problem (P1a). Let \( x_1^i = y_1^i \), \( i = 1, 2 \). Because Condition (43) holds, join \( p_2 \) and \( p_3 \) by a curve \( x_3^i \) such that \( \ddot{x}_1^i(t_2) = \dddot{x}_1^i(t_2) \) and that \( x_3^i \) has an \( L^\infty \) acceleration less than \( 2 \| a_L \| \). Note that the \( L^\infty \) acceleration of \( x_1 \), constructed in this way, is \( 2 \| a_L \| \). This, in turn, implies that any solution of Problem (P1b) must have \( \alpha \leq 2 \| a_L \| \). Therefore \( \alpha = 2 \| a_L \| \) for any solution of Problem (P1b).

Suppose that any solution of Problem (P1b) is given. Then \( x_1^i \) and \( x_3^i \) have minimum \( L^\infty \) accelerations \( \alpha^i \leq 2 \| a_L \| \), \( i = 1, 2 \). On the other hand, we have just established that, for any solution of Problem (P1b), \( \alpha = 2 \| a_L \| \). Therefore \( \alpha^i = 2 \| a_L \| \), \( i = 1, 2 \), and so \( x_1^i \) and \( x_3^i \) are quadratic in \( t \). Hence, indeed, \( \dot{x}_1(t_2) = v_L \). Since Condition (43) holds, there exists a solution curve segment \( x_3^i \) such that \( \dot{x}_3^i(t) < 2 \| a_L \| = \alpha \) for all \( t \in [t_2, t_3] \), and so, by Theorem 2(d), \( \lambda_3^i(t) \equiv 0 \).

**Remark 4** Note that if \( \| p_3 - p_5^i \| = (t_3 - t_2)^2 \| a_L \| \), then one does not necessarily have \( \lambda_3^i(t) \equiv 0 \). However, in this case, an interpolating curve can be constructed in exactly the same fashion as that outlined in the proof of Theorem 4: the resulting interpolating curve is quadratic in \( t \) with \( \alpha = 2 \| a_L \| \).

For the case of four points in \( \mathbb{R}^n \), Theorem 4 facilitates a test for determining whether or not the interpolating curve is singular, and its proof along with Remark 4 prescribes a technique for constructing a singular or quadratic nonsingular interpolating curve. Theorem 4 states the simple rule that if \( p_3 \) (resp. \( p_0 \)) is in the open ball with radius \( (t_3 - t_2)^2 \| a_L \| \) centered at \( p_5^i \) (resp. the open ball with radius \( (t_1 - t_0)^2 \| a_R \| \) centered at \( p_0^i \)), then the interpolating curve is singular and it is constructed in the way described in the proof of Theorem 4 and illustrated in Figure 1. Because of the function these balls serve we refer to them as *singularity test balls*. For points contained in the boundary of the ball, the test is inconclusive; however an interpolating curve in this case is constructed as a quadratic as pointed in Remark 4. If \( p_0 \) and \( p_3 \) are both outside the closure of their associated singularity test balls, respectively, then the interpolating curve is nonsingular.

Algorithm 1 given below describes a procedure for constructing a singular or quadratic nonsingular interpolating curve whenever either \( p_0 \) or \( p_3 \) is in a respective singularity test ball or its boundary (Steps 1 and 2). If neither \( p_0 \) nor \( p_3 \) is in the closure of a respective singularity test ball, then the algorithm also instructs (in Step 3) the way the nonsingular interpolating curve is constructed.
Algorithm 1

Step 0 Consider the points $p_0$, $p_1$, $p_2$ and $p_3$ in $\mathbb{R}^n$, which are given at $t_0$, $t_1$, $t_2$, $t_3$, respectively.

Step 1 (If $\|p_3 - p_0^L\| < (t_3 - t_2)^2 \|a_L\|$ then the interpolating curve is singular.) If $\|p_3 - p_0^L\| \leq (t_3 - t_2)^2 \|a_L\|$, the solution curve is given by

$$z(t) = \begin{cases} a_L t^2 + b_L t + c_L, & \text{if } t_0 \leq t \leq t_2, \\ \tilde{a} (t - t_2)^2 + v_L (t - t_2) + p_2, & \text{if } t_2 < t \leq t_3, \end{cases}$$

with

$$\tilde{a} = \frac{1}{(t_3 - t_2)^2} (p_3 - p_0^L),$$

where $a_L$, $b_L$, $c_L$, $v_L$ and $p_0^L$ are given in (31)-(35); stop.

Step 2 (If $\|p_0 - p_0^R\| < (t_1 - t_0)^2 \|a_R\|$ then the interpolating curve is singular.) If $\|p_0 - p_0^R\| \leq (t_1 - t_0)^2 \|a_R\|$, the solution curve is given by

$$z(t) = \begin{cases} \tilde{a} (t - t_1)^2 + v_R (t - t_1) + p_1, & \text{if } t_0 < t \leq t_1, \\ a_R t^2 + b_R t + c_R, & \text{if } t_1 < t \leq t_3, \end{cases}$$

with

$$\tilde{a} = \frac{1}{(t_1 - t_0)^2} (p_0 - p_0^R),$$

where $a_R$, $b_R$, $c_R$, $v_R$ and $p_0^R$ are given in (36)-(40); stop.

Step 3 If $\|p_0 - p_0^R\| > (t_1 - t_0)^2 \|a_R\|$ then the interpolating curve is nonquadratic and nonsingular. Therefore solve Problem (Pfd) and find $\alpha$, $\lambda_1^i$, $i = 1, \ldots, 4$. Substitute the values of $\alpha$ and $\lambda_1^i$ found into (29) to obtain $x_1^i(t)$, $t_{i-1} \leq t \leq t_i$, $i = 1, \ldots, 4$, the concatenation of which in turn gives $z(t) = x_1(t)$, $t_0 \leq t \leq t_N$. Stop.

4 Numerical Implementation and Examples

In this section, we provide three examples to illustrate implementation of the results presented in the previous section. Example 1 considers four specified points in $\mathbb{R}^2$, and so applies Algorithm 1 in parts (a), (b) and (c), where, in each part, the last point $p_3$ is taken to be different. Steps 1, 2 and 3 of Algorithm 1 are executed in parts (a), (b) and (c), respectively.

Example 2 is a slight variation of Example 1(c), where a fifth point in $\mathbb{R}^2$ is added to the previous four, giving rise to a nonsingular (and nonquadratic) interpolating curve.

Example 3 considers six points in $\mathbb{R}^3$ which results in another nonsingular (and nonquadratic) interpolating curve.

In Examples 1(c), 2 and 3, where the interpolating curves are nonsingular, Problem (Pfd) is solved by using Ipopt, version 3.2.4s, a popular optimization software based on an interior point method; see [37]. We use AMPL [17] as an optimization modelling language that employs Ipopt as a solver. We have performed all computations on a computer with the processor, Intel Core2 Duo CPU (i.e., two CPUs) at 1.4 GHz each, and with a 3-GB RAM.

It is common practice to approximate the state and control variables over a partition of the time horizon in optimal control problems, a process referred to as discretization, so that one can solve a finite-dimensional optimization problem, instead of an infinite-dimensional one, to obtain an approximate solution – see, for example, [4, 10, 11, 23, 24, 25]. We solve Problem (P3)
also by means of this direct discretization (discretize-then-optimize) approach, for comparison purposes. As in the case of Problem (Pfd), we employ the AMPL and Ipopt suite for the discretized problem.

**Example 1**

Suppose that $p_0 = (-1, 0)$, $p_1 = (0, 2)$, $p_2 = (1, 2)$, at $t_0 = 0$, $t_1 = 1/3$, $t_2 = 2/3$, respectively.

For each choice of $p_3$, we apply Algorithm 1.

(a) $p_3 = (2.9, 1)$: In this case, $x_1^3$ is found to be singular, because $p_3$ is contained by the singularity test ball of radius one centered at $p_c^3 = (2, 1)$, resulting in the solution in Step 1 of Algorithm 1. See the first graph in Figure 2 for an illustration. It can be verified that $x_1^1$ is not singular, because $p_0$ is not contained by the singularity test ball of radius $\sqrt{181}/20$ centered at $p_c^0 = (-0.55, 1.50)$ – see the second graph in Figure 2. The vector components of the resulting interpolating curve $z(t) = (z_1(t), z_2(t)) \in \mathbb{R}^2$ are simply determined to be

$$z_1(t) = \begin{cases} 3t - 1, & \text{if } 0 \leq t \leq 2/3, \\ 8.1t^2 - 7.8t + 2.6, & \text{if } 2/3 < t \leq 1; \end{cases}$$

$$z_2(t) = \begin{cases} -9t(t - 1), & \text{if } 0 \leq t \leq 2/3, \\ -3t + 4, & \text{if } 2/3 < t \leq 1. \end{cases}$$

In this case, the minimum $L^\infty$ acceleration is $\alpha = 18$. It should also be noted that, in fact, one has infinitely many solutions: $z(t)$ containing any curve segment $x_1^3$ with $16.2 \leq \|\ddot{x}_1\|_{L^\infty} \leq 18$ is a solution.

(b) $p_3 = (0, -1)$: Here, $x_1^3$ is found to be singular: $p_0$ is contained by the singularity test ball of radius $\sqrt{13}/2$ centered at $p_c^0 = (-2, 0.5)$, yielding the solution in Step 2 of Algorithm 1. See the second graph in Figure 3 for an illustration. It can be verified that $x_1^3$ is not singular, because $p_3$ is not contained by the singularity test ball of radius one centered at $p_c^3 = (2, 1)$ – see the first graph in Figure 3. The vector components $z_1(t)$ and $z_2(t)$ of the resulting interpolating curve
Figure 3: Example 1(b) with $p_3 = (0, -1)$. Here $x_1$ is singular, but $x_3$ is not.

Figure 4: Example 1(c) with $p_3 = (4, 1)$. Neither $x_1$ nor $x_3$ is singular.

$z(t)$ are easily determined to be

\[
\begin{align*}
z_1(t) &= \begin{cases} 
9t^2 - 1, & \text{if } 0 \leq t \leq 1/3, \\
-3(3t^2 - 4t + 1), & \text{if } 1/3 < t \leq 1; 
\end{cases} \\
z_2(t) &= \begin{cases} 
-3(t(3t - 5)/2, & \text{if } 0 \leq t \leq 1/3, \\
-27(t - 1)/2 - 1, & \text{if } 1/3 < t \leq 1. 
\end{cases}
\end{align*}
\]

In this case, the minimum $L^\infty$ acceleration is $\alpha = 9 \sqrt{13} \approx 32.4500$. As in (a), one actually has infinitely many solutions: $z(t)$ containing any curve segment $x_3$ with $18.5540 \approx 9 \sqrt{17}/2 \leq ||\bar{x}_1||_{L^\infty} \leq 9 \sqrt{13}$ is a solution.

(c) $p_3 = (4, 1)$: Here, neither $x_1$ nor $x_3$ is singular. The interpolating curve segment $x_1$ is not singular, because $p_0$ is not contained by the singularity test ball of radius $\sqrt{5}/2$ centered at $p_0 = (0, 3/2)$ – see the second graph in Figure 4 for an illustration. Similarly, $x_3$ is not singular, either, because $p_3$ is not contained by the singularity test ball of radius one centered at $p_3 = (2, 1)$ – see the first graph in Figure 4. As a result, Step 3 of Algorithm 1 is invoked.
We first solve Problem (Pfd) using AMPL and Ipopt to find
\[
\alpha = 20.762605861013
\]
\[
\lambda^1_1 = (-0.132870370719, -0.327962931793)
\]
\[
\lambda^2_1 = (1.000000000000, 0.027268487871)
\]
\[
\lambda^3_1 = (-0.867129629281, 0.300694439222)
\]
where the components of \(\lambda^i_1\), \(i = 1, 2, 3\), have been normalized (or scaled) by dividing them by the maximum absolute value of all six components, which in this case is \(|\lambda^2_{1,1}|\). Note that the solution for \(\lambda^i_1\) is unique up to a common positive scalar multiple. If \(\lambda^i_1\) are not normalized, with randomly generated initial guesses, each time one would very likely get a different solution for \(\lambda^i_1\).

Substituting these optimal values into (29) one obtains \(x^i_1(t)\), \(t_{i-1} \leq t \leq t_i\), the concatenation of which gives \(z(t) = x_1(t), t_0 \leq t \leq t_N\). The components \(z_1(t)\) and \(z_2(t)\) of the interpolating curve \(z(t)\) are depicted in the top left of Figure 5. By substituting the optimal values of \(\alpha\) and \(\lambda^i_1\), \(i = 1, \ldots, 4\), into (28) and (27), and concatenating, one also gets the velocity and acceleration curves, \(\dot{z}(t)\) and \(\ddot{z}(t)\), which are depicted in the middle top and top right of Figure 5, respectively. The graphs of the components of the multipliers (or adjoint or costate variables), \(\lambda_1(t)\) and \(\lambda_2(t)\), are also illustrated in the bottom left and middle bottom of Figure 5 for completeness. Finally, we show the trajectory of \(z(t)\) in \(\mathbb{R}^2\) in the bottom right of Figure 5. Although not shown graphically here, numerical solutions indeed reconfirm that \(\|\ddot{z}(t)\| = \alpha\), for all \(0 \leq t \leq 1\).

With the method we propose in this paper, i.e., by solving Problem (Pfd) instead of Problem (P3), we have achieved an accuracy of 12 decimal places for \(\alpha\) and the components of \(\lambda_1^i\) in less than 0.3 sec of CPU time. On the other hand, by direct optimization of the Euler discretization of the optimal control problem (P3), over a grid of 1800 discretization points, we could get \(\alpha\) correct to at most 4 decimal places (\(\alpha = 20.7626\)) in about 3.0 sec of CPU time. Since the solution state variables obtained are \(C^1\) (in particular, \(x_2(t)\) is only \(C^1\)), higher-order discretization schemes do not seem to help in refining the solution. For example, the trapezoidal rule, which is a second-order approximation, still results in \(\alpha\) to be correct to only four decimal places, with 1800 discretization time points. It takes about the same CPU time, 3.0 sec, as the Euler discretization does. One should recall that the trapezoidal rule, being a second-order approximation, requires the state variables to be at least \(C^2\) [18, 23].

With the Euler and trapezoidal approximations, the number of optimization variables are 10795 and 10800, respectively, as opposed to only seven variables to determine in Problem (Pfd). Despite the fact that direct discretization of Problem (P3) results in a sparse problem, an explosion in the number of variables makes the problem only much more difficult and time-consuming, not to mention the failure to meet the differentiability requirements in the case of the trapezoidal approximation.

Another clear advantage of our method is that, because we have closed form solutions in (27)-(29), we can evaluate the interpolation curve and its derivatives at any given \(t\). On the other hand, with a solution obtained by discretization, the (usually much poorer) approximate values of the curve and its derivatives are known only at the points of discretization. This means that, at points \(t\) other than the points of discretization, one would even need further interpolation!
Figure 5: Example 1(c), nonsingular interpolating curves, their derivatives, and associated multiplier (or adjoint, or costate) curves.
Example 2 Consider the points \( p_0 = (-1, 0) \), \( p_1 = (0, 2) \), \( p_2 = (1, 2) \), \( p_3 = (4, 1) \) and \( p_4 = (4, 0) \) given at \( t_0 = 0 \), \( t_1 = 1/4 \), \( t_2 = 1/2 \), \( t_3 = 3/4 \) and \( t_4 = 1 \). Now that we have five points, Algorithm 1 is not applicable anymore. However, we recall Proposition 4 and assume that there exists a solution to Problem (Pfd).

We solve Problem (Pfd) using AMPL and Ipopt to find

\[
\begin{align*}
\alpha &= 55.935918218882 \\
\lambda_1^1 &= (-0.019181930951, -0.017508694823) \\
\lambda_2^1 &= (0.396919631617, -0.070361956619) \\
\lambda_3^1 &= (-1.000000000000, 0.142414830069) \\
\lambda_4^1 &= (0.622262299334, -0.054544178627)
\end{align*}
\]

Figure 6 depicts the graphs of the interpolating curve, along with its velocity and acceleration curves, as well as the graphs of the adjoint variables.

As in the case of Example 1(c), by solving Problem (Pfd) we have achieved an accuracy of 12 decimal places for \( \alpha \) and the components of \( \lambda_i^1 \) in about 0.3 sec of CPU time. On the other hand, by direct optimization of the Euler discretization of the optimal control problem (P3), over a grid of 2000 discretization points, we could get \( \alpha \) correct to at most four decimal places in about 3.0 sec of CPU time. As in Example 1(c), the trapezoidal rule, does not improve the accuracy of \( \alpha \).

In this example, the number of optimization variables used with Euler discretization is 11995, as opposed to only nine variables to determine in Problem (Pfd).
Figure 6: Example 2, nonsingular interpolating curves, their derivatives, and associated multiplier (or adjoint, or costate) curves.
Example 3 Consider \(p_0 = (1, 0, 3), \ p_1 = (1, 1, 3), \ p_2 = (3, 3, -1), \ p_3 = (3, -1, -2), \ p_4 = (-1, 1, 0), \ p_5 = (2, 4, 3)\) given at \(t_0 = 0, \ t_1 = 1/5, \ t_2 = 2/5, \ t_3 = 3/5, \ t_4 = 4/5, \ t_5 = 1.\) This time we have six points given in \(\mathbb{R}^3.\) As in Example 2, we invoke Proposition 4 and assume that there exists a solution to Problem (Pfd) with these given points.

We solve Problem (Pfd) using AMPL and Ipopt to find

\[
\begin{align*}
\alpha & = 241.3193402320 \\
\lambda_1^1 & = (0.018177821, \ 0.062302851, \ -0.051273586) \\
\lambda_1^2 & = (-0.019455171, \ -0.473702614, \ 0.168980653) \\
\lambda_1^3 & = (-0.409886118, \ 1.000000000, \ 0.039190136) \\
\lambda_1^4 & = (0.779721907, \ -0.683683770, \ -0.153457943) \\
\lambda_1^5 & = (-0.368558439, \ 0.095083533, \ -0.003439259)
\end{align*}
\]

Figure 7 depicts the graphs of the interpolating curve, along with its velocity and acceleration curves.

This time, by solving Problem (Pfd) we have achieved an accuracy of 10 decimal places for \(\alpha\) in about 0.4 sec of CPU time. The components of \(\lambda_i^1,\) on the other hand, are accurate up to nine decimal places. However, by direct optimization of the Euler discretization of the optimal control problem (P3), over a grid of 1200 discretization points, we could get \(\alpha\) correct to at most two decimal places in about 80 sec of CPU time.

In this example, the number of optimization variables used with Euler discretization is 15591, as opposed to only 16 variables to determine in Problem (Pfd).
Figure 7: Example 3, nonsingular interpolating curves, their derivatives, and associated multiplier (or adjoint, or costate) curves.
5 Discussion and Conclusion

We have characterized interpolating curves minimizing the pointwise maximum length of their acceleration in the Euclidean space, $\mathbb{R}^n$, by utilizing the simple but effective tools of optimal control theory. Using optimal control terminology, we have classified interpolating curves as singular and nonsingular, and studied them further.

In the scalar case, i.e., when $n = 1$, a nonsingular interpolant is the well-known perfect spline: it is piecewise-quadratic and the absolute value of its acceleration is constant. In the more general case when $n \geq 2$, although the acceleration of a nonsingular interpolating curve has a constant length throughout, the curve is not piecewise quadratic anymore; it is not even piecewise polynomial. We derived analytical expressions for this type of curves in terms of elementary functions of a finite number $(nN + 1)$ of unknown constants which can be efficiently computed by solving a relatively small-scale optimization problem (Theorem 3 and Problem (Pfd)).

We devised a test which can be used to conclude whether (a segment of) an interpolating curve is singular, or nonsingular, in the case of four points, i.e., when $N = 3$, in Theorem 4, and implemented this test in Algorithm 1. Although the case of four points might seem a little restrictive, the test itself is novel even in the case when $n = 1$. We showed some example instances as to how singular interpolating curves can be constructed. The rule of construction we have described can possibly be extended to cases with more than four points, as part of future work, by dealing with configurations of singular and nonsingular segments in a fashion similar to that provided in Theorem 4 and its proof, and the ensuing Algorithm 1.

We have illustrated by means of examples that our approach is a significant improvement over the direct so-called discretize-then-optimize approach. Our numerical approach uses exact analytical expressions for a nonsingular interpolating curve, an approach which only needs a relatively small number of constant unknowns to be determined. Our approach does not need discretization, and as a result, it is free from the curse of dimensionality. Another viable numerical approach would have been to use an indirect technique instead of a direct one, which involves solving a two-point boundary-value problem arising from the optimality conditions (or the maximum principle) for Problem (P4). This technique would possibly have incorporated shooting methods. These kinds of methods are well-known to suffer from poor initial guesses and accumulation of error. Such drawbacks are not of concern for our approach, either.

References


