A Class of Models and Budget Strategies for the Control of Heroin Epidemic in Australia*

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Abstract

A class of control models is introduced for the heroin epidemic in Australia, for the period spanning 1986 to 1997. The models are reduced to two representative models capturing, respectively, the most conservative and the most optimistic assumptions about the effectiveness of budget expenditures on the population of heroin users. These analyses are performed under three alternative scenarios: Zero budget or uncontrolled scenario, stabilising scenario, and steady improvement scenario. The population growth under the zero budget scenario is viewed as a benchmark for assessing the other two scenarios. For the latter, optimal budget strategies are also derived, minimizing a trade-off function of the number of users and the budget. Collectively, the user population trends induced by these budget strategies will help decision makers assess the likely impact of expenditures under a wide range of conditions. Comparisons and interpretations of these findings are included.

Key words: Heroin users, drug epidemics, dynamic modelling, optimal control.

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1 Introduction

Illicit drug use in Australia, as well as around the world, has exacted significant costs on societies and governments. The problem has reached epidemic proportions in most industrialized countries, despite widespread counter-measures and campaigns organized at national and international levels. Staines [21] states that drugs have become a “global habit.” Governments announce (often controversial) budget allocations for prevention, treatment and law enforcement programs to fight the problem. Therefore it is important to develop an understanding of how the prevalence and trends of drug use interact with the amount of funding allocated to such programs. This requires construction and use of quantitative models.

Drug epidemics change course over time, that is, the users’ prevalence and trend vary from year to year. Consequently, epidemics are often referred to as dynamic. Seminal dynamical models for drug epidemics include those of Everingham & Rydell [8], Behrens et al. [4, 5], and Tragler et al. [22]. Their models emulate behaviour of the cocaine epidemic in the United States of America. Everingham & Rydell propose a discrete-time model, where the dynamic (or state) variables are the so-called light and heavy users, and the process is assumed to be governed by Markovian assumptions. Behrens et al. [4, 5] present a continuous-time dynamical control model with the same state variables as in [8] and the controls being the treatment and prevention spending. This model has led to further work including the study in [11]. Tragler et al. [22] model the same epidemic by a single state equation where the controls are incorporated in the form of the treatment and law enforcement spending. More generally, mathematical modelling and analysis of illicit drug use is now an active research area and includes works focusing on the dynamic interactions between law enforcement and drug markets [1, 2, 3, 9, 14].

While the cocaine use appears to be a major illicit drug problem in U.S.A., Australia is geographically and historically linked more to the growing pandemic of global heroin use. Australia was the largest per capita consumer of heroin among industrialized countries during the 1930s and in 1953 [6]. In more recent times (since the 1970s) Australia had a widely recognised ‘heroin problem’, that became very severe from the mid 1980s onwards. In 1998, seven hundred and thirty-seven heroin overdose deaths were reported [6], with prevalence estimates of dependent users in 1997 given at around 75,000 country-wide [10]. Subsequently, heroin use is considered to be the most important illicit drug problem in Australia because of its alarming prevalence, rate of growth and the consequent social cost.

In this paper we construct a simple class of dynamical control models for the heroin epidemic in Australia. To develop these models we use data derived in Kaya et al. [12], namely data representing the time history of the number of heroin users and the budget expenditure over the period 1986-1997. Given the inherent uncertainty in data
and the relatively short period of time it represents, we make as few assumptions as we can. For the same reasons we focus our investigation and present results only for the time period covered by the data.

First, we introduce the class of models based on the budget and heroin user data. Using an intuitive argument, these models are reduced to two representative models: Model A and Model B. These two models capture, respectively, the most conservative and the most optimistic assumptions about the effectiveness of budget expenditures on the population of heroin users. Subsequent analyses are performed under three alternative scenarios:

1. Zero budget or uncontrolled scenario that, as the name indicates, assumes no expenditures on the prevention and treatment programs,

2. Stabilising scenario that maintains the number of users at a constant level from 1986 onwards,

3. Steady improvement scenario that achieves a linear decrease of 5% in the number of users.

The population growth under the zero budget scenario is viewed as a benchmark for assessing the other two scenarios. For the latter, optimal budget strategies are also derived, minimizing a trade-off function of the number of users and the budget. Collectively, the user population trends induced by these budget strategies will help decision makers assess the likely impact of expenditures under a wide range of conditions. Comparisons and interpretations of these findings are included.

2 Modelling

In this section, first we present the data we use, and then introduce a class of models and find best fits for these models.

2.1 Data

A time history of the number of heroin users in Australia has recently been reported by Kaya et al. (2001), which is depicted in Figure 1. The historical data have been reconstructed by using the 1998 National Drug Strategy Household Survey (NDSHS '98).

Figure 2 shows the combined treatment and prevention budget allocated by the governments in Australia to fight drug abuse, between the years 1986 and 1997 [15, 16]. Spending on the law enforcement is not incorporated in this study.
Figure 1: Time history of the number of heroin users in Australia.

Figure 2: Combined budget for treatment and prevention.
2.2 A Class of Models and Envelopes

The historical data in Figures 1 and 2 suggest that the heroin use in Australia is on a sharp increase, which has recently reached epidemic proportions, despite the efforts made (in terms of budget expenditure) to control this upward trend.

The budget data are available only over the period between 1986 and 1997. Therefore we aim to model the behaviour of the epidemic over the period spanning these years. We want to establish a model that would explain some of the essential features of the phenomenon, and yet remain simple. We identify two essential qualitative features to represent in the model:

(i) an increase in the budget leads to a decrease in the number of users,

(ii) Drug user population tends to grow exponentially, if there is very small (or no) budget spending.

The following model possesses these features,

\[ N(t) = \left( a + be^{\alpha t} \right) F(B(t)), \]

where the variable \( N(t) \) is the number of users in year \( t \), the function \( F(B(t)) \) represents the control effort, and \( B(t) \) denotes the combined prevention and treatment budget. In (1), \( a, b \) and \( \alpha \) are constant parameters.

The control effort function must also satisfy certain requirements. In particular, \( F(B(t)) \equiv 1 \) needs to correspond to the case when there is no expenditure, that is, when \( B(t) \equiv 0 \), which we refer to as the zero budget or uncontrolled case. We denote by \( N_{\text{uc}}(t) := (a + be^{\alpha t}) \) the number of users when no budget is spent on the problem. Note that in this case, \( \alpha \) accounts for the rate of the uncontrolled growth of the epidemic, while \( a \) and \( b \) measure the prevalence. Another desirable property of the control effort, albeit technical, is that as \( B(t) \rightarrow \infty \), \( F(B(t)) \rightarrow 0 \). This property reflects the “belief” that with unlimited resources devoted to the heroin problem, the population of users could be reduced to zero.

In particular, we consider the form

\[ F(B(t)) = e^{-\beta (c_1 B(t) + c_2 B(t-1))} \]

where \( c_1 \) and \( c_2 \) are prescribed positive constants. Suppose the time \( t \) represents the current year. Then the form in (2) tells us that this year’s and last year’s budget will have a combined effect to reduce this year’s number of users. The constants \( c_1 \) and \( c_2 \) determine the relative importance of each year’s budget in this combination. The parameter \( \beta \) in (2) may be interpreted as the effectiveness of the budget.

With the prescribed values of \( c_1 \) and \( c_2 \), the values of the parameters \( a, b \) and \( \alpha \) in (1), and \( \beta \) in (2), can be found, such that the model matches the historical data in
Figures 1 and 2, in a least-squares sense. We have used the statistical software SPSS to determine these parameters. The Levenberg-Marquardt method (see, [13]) was chosen to solve the resulting equations in the nonlinear regression.

For computational and bookkeeping reasons, the number of users $N(t)$ is taken to be in thousands, and the budget $B(t)$ in millions of dollars. The time $t = 16$ corresponds to the initial year 1986, and $t = 27$ to the final year 1997.

Consider the following models with different values of $c_2$, where $c_1$ is normalised as 1, except in Model 9 where $c_1 = 0$.

<table>
<thead>
<tr>
<th>Model</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$c_2$</td>
<td>0</td>
<td>1/5</td>
<td>1/3</td>
<td>1/2</td>
<td>1</td>
<td>3/2</td>
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</tbody>
</table>

The resulting zero-budget ($B(t) \equiv 0$) behaviour obtained for each model is depicted in Figure 3. The figure shows that if no money were spent on the problem, then with Model 1 (where $c_1 = 1$, $c_2 = 0$), by the year 1997 the heroin user population would have reached 200,000. On the other hand Model 9 (where $c_1 = 0$, $c_2 = 1$) suggests that the population would have grown to a much lower level of 118,000.

It is interesting to note that the zero-budget trajectories of all the other models lie between the trajectories of Models 1 and 9. In other words, Model 1 appears to form an upper envelope, while Model 9 forms a lower envelope, for the zero-budget behaviour of all models with arbitrary positive constants $c_1$ and $c_2$. From now on we will call Models 1 and 9 respectively as Models A and B, for convenience.

From the behaviour displayed in Figure 3, Model A can be regarded as optimistic, because it considers the budget that has been allocated to the problem in the past years (between 1986 and 1987) to be quite effective. It states that if no money had been spent in the past, then the number of heroin users would have been about twice the actual number in 1997. Model B, on the other hand, represents a rather pessimistic viewpoint. It states that even if no money were spent in the past, the size of the problem would have hardly been any different from the size in 1997. The zero-budget trajectory for Model B is only slightly above the historical data as can be seen in Figure 3. It suggests that a significantly larger budget should have been allocated in the past, in order to stop the upward trend of the number of heroin users.

Since Models A and B represent the two extreme situations, we will give our analysis using these two models only, in the rest of the paper. Model A can be rewritten as

$$N(t) = \left(a + be^{\alpha t}\right) e^{-\beta B(t)}$$

and Model B as

$$N(t) = \left(a + be^{\alpha t}\right) e^{-\beta B(t-1)}$$
Figure 3: What various models say if no budget is spent. The historical data is shown by ◆.

Model A may be referred to as memoryless, because both the system variable \( N(t) \) and the input \( B(t) \) are expressed at time \( t \). However, in Model B the value of \( N(t) \) depends on the previous year’s budget \( B(t - 1) \). In other words, the effect of last year’s budget becomes visible in the current year. The values of the parameters for both models have been found by least-squares fit as

\[
\begin{array}{c|cccc}
\text{Model} & a & b & \alpha & \beta \\
\hline
\text{Model A} & 66.8955 & 0.105636 & 0.264200 & 0.0118935 \\
\text{Model B} & 50.6000 & 0.0138604 & 0.314354 & 0.00274755
\end{array}
\]

It should be noted that even though we have the historical data (trends) for the Australian heroin user population running from 1971 onwards [12], budget figures are only available from 1986 onwards. Due to this reason, we restricted the fitting of the models to the time duration between 1986 and 1997.

Figure 4 illustrates the modelled (or fitted) number of users for Models A and B, along with the observed (or historical) number of users. The fit for Model B starts from 1987, as the number of users in 1986 depends on the budget in 1985, which was
2.3 Sensitivities

Because of the inherent uncertainties in data, a fitted model needs to be tested to see whether it would still reasonably represent the real process whose output is usually different than the available data. One way of doing this test is to check the sensitivity of a model to small changes in its parameters. In our case we can carry out this task by considering the deviations of the model (dependent) variable $N(t)$, from its fitted nominal values, resulting from changes in the values of its parameters, $a$, $b$, $\alpha$, and $\beta$.

Percentage deviations of $N(t)$ from its nominal value each year are depicted in Figures 5(a)-(d) corresponding to a 1% change in each parameter of Models A and B.

In Figure 5(a), we see that a 1% change in $a$ results in less than 1% change in the number of users $N(t)$ for both models throughout the time period 1986-1997. Given the conceivably large uncertainties in the data, this amount of variation can be considered reasonable. Similarly a 1% change in $b$ results in less than 0.7% change in $N(t)$, as can be seen in Figure 5(b). The variations in $N(t)$ are also reasonable with respect to a 1% change in $\beta$, which is less than 0.7%, as shown in Figure 5(d). Note that in this case, the variations in $N(t)$ for Model B are remarkably smaller than those for Model A. Because $\beta$ is a measure of the efficiency of the budget, this observation is in line with our earlier comment that in Model B the budget is less able to affect the number of users.
Figure 5: Percentage deviations of $N(t)$ from its nominal value resulting from a 1% change in the parameters (a) $a$, (b) $b$, (c) $\alpha$, and (d) $\beta$. 
The deviation 0.5 - 5% of $N(t)$ from the nominal values for a 1% change in $\alpha$, as shown in Figure 5(c), is not insignificant. However it must be noted that $\alpha$ represents an exponential growth rate. Therefore a small change in $\alpha$ is magnified in the number of users as time $t$ grows.

3 Direct Budget Strategies

Equations (3) and (4) can be directly solved for the respective budgets, $B(t)$, for each model. One obtains for Model A,

$$B(t) = \frac{1}{\beta} \ln \left( \frac{a + b e^{\alpha t}}{N(t)} \right), \quad (5)$$

and, for Model B,

$$B(t) = \frac{1}{\beta} \ln \left( \frac{a + b e^{\alpha(t+1)}}{N(t+1)} \right). \quad (6)$$

For both models, different ‘direct’ scenarios regarding the number of heroin users and the budget spending can now be investigated. Three of these are discussed below.

3.1 Stabilising Scenario

What budget expenditure would one require in order to maintain the number of users at a constant level from, say, 1986 onwards? The prevalence in 1986 is around 49,000. Direct calculations using (5) and (6) yield the budget profiles depicted in Figure 6(a) for Models A and B.

According to Model A, an extra $215$ million expenditure on top of the $570$ million already spent between 1986-1997 could have kept the number of heroin users at 49,000. According to Model B, the necessary extra spending would require $925$ million on top of $512$ million already spent over the period 1986-1996.

3.2 Steady Improvement Scenario

What budget expenditure would have been necessary to achieve a linear decrease with a 5% slope from 1986 onwards? This decrease constitutes a reduction of, roughly, 2,450 in the number of users each year. Direct calculations using (5) and (6) yield the budget profiles depicted in Figure 6(b) for Models A and B.

According to Model A, an extra $570$ million expenditure on top of $570$ million already spent between 1986-1997 could have reduced the number of heroin users from
49,000 in 1986, to around 22,000 in 1997. According to Model B, the necessary extra spending is $2,460 million on top of $512 million\(^1\) already spent during 1986-1996, which is remarkably higher than that suggested by Model A. This is again in line with the ‘pessimistic’ nature of Model B.

### 3.3 Zero Budget or Uncontrolled Scenario

Apart from the above two scenarios, we also considered the uncontrolled scenario. That is the scenario that captures how the user population would have grown if there were no budget spending on the drug problems. The details are already given in Section 2.2, see also Figure 3. According to Model A the user population would have reached 200,000 mark in 1997 while it would have become 117,000, also in 1997, according to Model B.

### 4 Optimal Budget Strategies

In the preceding section we derived budget strategies with the aim of achieving a prescribed population profile. However, these strategies do not necessarily satisfy any standard optimality criterion. In this section we find control strategies in an optimal way, with respect to a specified criterion.

It is desirable to reduce the size of the user population to the lowest possible value, subject to the limitations on the budget. In this study we do not consider prescribed

\(^{1}\)Note that the discrepancy between $570 million and $512 million results from the fact that Model B is based on data spanning one year less than that for Model A.
constraints on the budget; however, we consider choosing as small a budget as possible, to achieve the smallest possible population over the time horizon of concern. Relative sizes of the minimum number of users and minimum budget depend on the relative importance we place on these two "criteria." As a result we consider minimizing a weighted sum of the number of users and budget, where the weights reflect their relative importance.

Define the state variable \( x(t) := N(t) \) and the control function \( u(t) := B(t) \), which are common notation in the control systems literature. This notation will particularly make the appearance of calculations given in Appendix A neater and more standard. For each of the models A and B, we consider two different formulations. Model A is treated as a continuous-time system, while Model B as a discrete-time system. In each problem the concern is to minimize a weighted sum of the number of users and budget, albeit in the different settings of continuous- and discrete-time associated with the respective models.

Note that we aim to minimize the number of users but at the same time keep an eye on the budget spending! With this aim we pose two kinds of optimization problems for each model. The problems for Model A are stated as follows.

\[
\text{(PA1)} \begin{cases} 
\text{minimize} & \int_{t_0}^{t_f} \left( \alpha_1 x(t) + \alpha_2 u(t) \right) dt \\
\text{subject to} & x(t) = \left( a + be^{\alpha_1 t} \right) e^{-\beta u(t)},
\end{cases}
\]

and

\[
\text{(PA2)} \begin{cases} 
\text{minimize} & \int_{t_0}^{t_f} \left( \alpha_1 x^2(t) + \alpha_2 u^2(t) \right) dt \\
\text{subject to} & x(t) = \left( a + be^{\alpha_1 t} \right) e^{-\beta u(t)},
\end{cases}
\]

where the initial and final times \( t_0 \) and \( t_f \) are specified. In particular, \( t_0 = 16 \) (year 1986) and \( t_f = 27 \) (year 1997). By choosing the coefficients \( \alpha_1 \) and \( \alpha_2 \), more precisely the ratio \( \alpha_1 / \alpha_2 \), we can place importance either on the number of users \( x(t) \) or on the budget \( u(t) \). Note that the coefficients \( \alpha_1 \) and \( \alpha_2 \) capture the "importance weights" assigned to the number of users and the budget, respectively. Thus the ratio \( \alpha_1 / \alpha_2 \) represents the number of users (in thousands) per unit of budget (in million dollars) that we shall call the "user per dollar ratio".

Although the cost expressions in Problems (PA1) and (PA2) represent related optimization criteria, their solutions will be seen to be quite different. Our aim in posing both formulations is to extract as much information about the nature of the underlying social problem as possible.

Necessary optimality conditions for Problem (PA1) yield the following simple control law. Details of the derivations are given in Appendix A.
\[ u(t) = \frac{1}{\beta} \ln \left( \frac{\alpha_1}{\alpha_2} \beta (a + b e^{\alpha t}) \right). \]  

(7)

Furthermore, it turns out that the optimal population variable \( x \) is simply a constant depending on the ratio \( \alpha_1 / \alpha_2 \) and \( \beta \), the efficiency of the budget. Namely,

\[ x(t) = \frac{\alpha_2}{\alpha_1 \beta}. \]  

(8)

It is interesting to note that with the choice \( N(t) = \alpha_2 / (\alpha_1 \beta) \), the direct budget \( B(t) \) given in (5) coincides with (7), the optimal budget strategy for (PA1).

The solution of Problem (PA2) is obtained as

\[ u(t) = \frac{\alpha_1}{\alpha_2} \beta x^2(t), \]  

(9)

for the details of which, see Appendix A. In this case the optimal control is directly proportional to the square of the number of users. Recall that the factor \( \alpha_1 / \alpha_2 \) denotes the relative importance of the number of users over the budget. In simple terms, if one wishes to place “twice” more importance on the number of users, then the optimal budget will have to be doubled.

It can also be shown that

\[ u(t) = \frac{1}{2\beta} L_w \left[ \frac{2\alpha_1}{\alpha_2} \beta^2 \left( a + b e^{\alpha t} \right)^2 \right], \]  

(10)

where \( L_w \) stands for the Lambert w function [7], that is the solution of the equation

\[ w e^w = \frac{2\alpha_1}{\alpha_2} \beta^2 \left( a + b e^{\alpha(k+1)} \right)^2 \]

for \( w \).

The problems for Model B can be stated similarly, albeit using the discrete time variable \( k \), that is,

(PB1) \[
\begin{align*}
\text{minimize} & \sum_{k=k_0}^{k_f} (\alpha_1 x(k) + \alpha_2 u(k)) \\
\text{subject to} & x(k) = \left( a + b e^{\alpha k} \right) e^{-\beta u(k-1)}
\end{align*}
\]

and

(PB2) \[
\begin{align*}
\text{minimize} & \sum_{k=k_0}^{k_f} (\alpha_1 x^2(k) + \alpha_2 u^2(k)) \\
\text{subject to} & x(k) = \left( a + b e^{\alpha k} \right) e^{-\beta u(k-1)}
\end{align*}
\]
where the initial and final times \( k_0 \) and \( k_f \) are specified. In particular, \( k_0 = 17 \) (year 1987), \( k_f = 27 \) (year 1997), and that \( x(17) = 49,000 \).

Necessary optimality conditions for Problem (PB1) yield the following control law, namely, an optimal budget policy:

\[
u(k) = \frac{1}{\beta} \ln \left( \frac{\alpha_1}{\alpha_2} \beta \left( a + be^{\alpha(k+1)} \right) \right), \tag{11}
\]

for \( k = 17, \ldots, 26 \). Again, as in the case of (PA1), optimal \( x \) is simply a constant

\[
x(k) = \frac{\alpha_2}{\alpha_1 \beta} \tag{12}
\]

and the direct control given by (6) coincides with the optimal control (11), as long as we consider the constant user populations. It is, perhaps, significant that in both (PA1) and (PB1) the use of optimal budget control leads to a constant population of users, see (8) and (12).

On the other hand, the solution to Problem (PB2) is given by

\[
u(k) = \frac{\alpha_1}{\alpha_2} \beta x^2(k+1) \tag{13}
\]

for \( k = 17, 18, \ldots, 26 \). While the above expression appears counter-intuitive, it is a direct consequence of the fact that in Model B the user population in a given year depends on the budget expenditure from the previous year, see (4). Alternatively, in terms of the Lambert \( w \) function

\[
u(k) = \frac{1}{2\beta} L_w \left[ \frac{2\alpha_1}{\alpha_2} \beta^2 \left( a + be^{\alpha(k+1)} \right)^2 \right]. \tag{14}
\]

For the evaluation of the Lambert \( w \) function in (10) and (14), MATLAB has been used.

For different values of \( \alpha_1/\alpha_2 \) in each of the formulations, the optimal budget profiles over the period 1986-1997 and the corresponding trajectories for the number of users have been obtained. These are depicted in Figure 7 for the linear cost formulation, (i.e. for Problems (PA1) and (PB1)) and in Figure 8 for the quadratic cost formulations (i.e. for Problems (PA2) and (PB2)). The ratios \( \alpha_1/\alpha_2 = 1, 10, \) and 100 are considered for Problems (PA2) and (PB2). These ratios give increasingly more emphasis to minimizing the number of users than minimizing the budget. For the Problems (PA1) and (PB1)) \( \alpha_1/\alpha_2 = 7, 10, \) and 100 are considered. In this case the reason for taking \( \alpha_1/\alpha_2 = 7 \) instead of 1 is that the solution to Problem (PB1) does not exist for the ratio \( \alpha_1/\alpha_2 = 1 \). In the next two subsections we give a summary of the story told by Figures 7 and Figures 8.
Figure 7: Optimal profiles for the number of users (in thousands) and budget (in million dollars) according to PA1 and PB1: [(a), (b)] $\alpha_1/\alpha_2 = 7$; [(c), (d)] $\alpha_1/\alpha_2 = 10$; [(e), (f)] $\alpha_1/\alpha_2 = 100$. 
Figure 8: Optimal profiles for the number of users (in thousands) and budget (in million dollars) according to PA2 and PB2: [(a), (b)] $\alpha_1/\alpha_2 = 1$; [(c), (d)] $\alpha_1/\alpha_2 = 10$; [(e), (f)] $\alpha_1/\alpha_2 = 100$. 
4.1 Numerical Results with the Linear Cost formulation

Case I ($\alpha_1/\alpha_2 = 7$). We place a slightly higher order of importance on the user numbers to the relative budget. As a result, in the earlier years Model B proposes similar magnitude of budget to those actually spent. However, in years 1992 onwards the budget recommended by Model B is a little higher than actual expenditures. Overall $710$ million extra spending is required on top of $480$ million already spent over the period 1987-96. This budget regime results in a constant level of 52,000 users throughout the time horizon.

On the other hand, the optimal strategy with Model A appears to be quite different. The optimal budget strategy (7) for (PA1) results in a constant population of 12,000 users throughout, which is much lower than population of 52,000 users resulting from (PB1). Furthermore, in (PA1) this is achieved with an almost steady budget ranging between 150 and 235 million dollars per year, see Figure 7(a)-7(b).

Case II ($\alpha_1/\alpha_2 = 10$). Here we place even higher importance on the number of users relative to the budget. The optimal strategy in (PB1) requires further funds that are in the range of 200 to 400 million dollars every year in the years after 1990. This keeps the number of users at the level of 36,000. The (PA1) formulation, on the other hand, yields an optimal strategy (7) where the budget expenditure is again almost steady and is between 180 and 260 million dollars every year but this time with a number of users at the constant level of 8,400 (see Figure 7(c)-7(d)).

Case III ($\alpha_1/\alpha_2 = 100$). This is a case when we place an extremely high importance on the number of users relative to the budget. Both formulations give optimal strategies requiring significantly larger budgets. In the case of (PA1), the proposed budget is in the range $380-450$ million every year. However, in the case of (PB1) huge amounts of $1000-1275$ million every year are required. On the other hand, the population of users resulting from these expenditures drop to the "tolerable" level of 3,600 and 840, respectively, see Figure 7(e)-7(f).

4.2 Numerical Results with the Quadratic Cost formulation

Case I ($\alpha_1/\alpha_2 = 1$). A similar importance is placed on the user numbers and the budget. Consequently, Model B proposes a much lower budget than that actually spent. This budget results in only slightly higher number of users. The optimal strategy with (PA2) is almost the same as the historical one. In recent years, however, it requires a higher budget, which in turn results in 20% lower number of users in 1997. See Figure 8(a)-8(b).

Case II ($\alpha_1/\alpha_2 = 10$). We place a higher importance on the number of users relative to the budget. The optimal strategy with formulation (PA2) requires $720$ million extra spending on top of $570$ million already spent over the period 1986-97. This
keeps the number of users between around 25,000 and 35,000. Formulation (PB2), on the other hand, results in an optimal strategy where the extra budget expenditure is around $330 million on top of $480 million already spent during 1987-96, with a number of users profile almost the same as the historical one, except that the number of users in 1997 is around 30% lower, (see Figure 8(c)-8(d)).

Case III ($\alpha_1/\alpha_2 = 100$). Since we place a much higher importance on the number of users relative to the budget, both formulations yield optimal strategies requiring significantly higher funds. In the case of (PA2), the necessary extra spending is $1,600 million, and in the case of (PB2), this is $2,260 million. The number of users resulting from these expenditures are lower than 40,000 throughout the period. In the case of (PA2), in particular, the number of users are merely between 10,000 and 15,000. (Figure 8(e)-8(f)).

5 Discussion and Conclusion

In this paper a class of control models representing heroin epidemic in Australia has been studied. The models were developed with help of data capturing both the evolution of the number of heroin users and the Australian government’s budget spending on drug problems. This budget spending enters the models as the single control function, where the number of users is the state variable. In our study the budget consists of the prevention and treatment components. Spending on law enforcement is not taken into account.

The models constructed have their limitations, because the internal dynamics remain unexplored. These involve the issues such as the influence of prevention and treatment components of the budget on the dynamics of dependent and light heroin users, interactions of long- and short-term users, or users in different age groups. Nevertheless, the contribution of this class of models is due to the fact that they provide a summary of the problem and an understanding about the amount of budget needed to achieve a 'desired' evolution of the population of the heroin users.

It should be noted that each of the models in the class provide an excellent fit to the observed heroin data over the period of consideration. However, it is the uncontrolled scenario, as shown in Figure 3, which reveals the true picture of the intrinsic nature of each model in the class. Consequently, the 'most optimistic' and the 'most pessimistic' models among the class, namely Model A and Model B are identified in the process. It is therefore sensible to choose only these two models for further analysis and study.

Effectively, three direct scenarios are considered: (i) Zero-budget or, uncontrolled, scenario, (ii) Stabilising scenario, that is, one that requires budget spending that will maintain constant population of heroin users, and (iii) Steady improvement scenario that requires budget spending levels that will achieve linear decrease in the population
of heroin users. Optimal budget strategies are also derived, minimizing a trade-off function of the number of users and the budget. Collectively, the user population trends induced by these budget strategies will help decision makers assess the likely impact of expenditures under a wide range of conditions. Comparisons and interpretations of these findings are included.

As can be seen from the results regarding the stabilising scenario, the heroin user population could be kept at a level of 49,000 by an additional spending in the range of $\$215 - 925$ million on top of $\$570$ million already spent. Of course, the lower limit of budget $\$215$ million is suggested by Model A, while the upper limit is according to Model B.

It is interesting to discover later that these budget profiles turn out to be optimal according to the optimization criteria of formulation, (PA1) and (PB1) for Model A and Model B, respectively, see Section 4. More precisely, if the user per dollar\(^1\) ratio $\alpha_1/\alpha_2 = 1.7159$ in (PA1), we obtain the same budget strategy as mentioned here according to Model A. On the other hand, the user per dollar ratio $\alpha_1/\alpha_2 = 7.4278$ in (PB1) yields the same budget profile according to Model B. However, we must note that in order to achieve the same constant user profile of 49,000 we need to increase the $\alpha_1/\alpha_2$ ratio in (PA1) by a factor of approximately four, which is obviously due to the pessimistic nature of Model B.

Strategies obtained with the formulations (PA2) and (PB2) yield entirely different results from the ones obtained with the above formulation. The feedback laws suggest that an optimal control is directly proportional to the square of the number of users.

When we give a similar order of importance to the number of users and the budget, Model A, yields almost the same budget strategies as actually spent except in the late 90's, which in turn results in 20% lower number of users in 1997. Model B, on the other hand, yields a much lower budget than actually spent which results in only a slightly higher user profile in the years after 1991.

Indeed, if we give 10 times higher importance to the budget than the number of users, ($\alpha_1/\alpha_2 = 0.1$), Model B simply yields an essentially nil budget, with a user profile only slightly higher than the historical one, which is not surprising, as according to Model B the budget is not efficient in bringing down the user population.

For the ratio $\alpha_1/\alpha_2 = 10$, that is, when we place 10 times higher importance over the number of users, the budget resulting from Model A is higher, but it keeps the number of users at a much lower level, between 25,000 and 35,000, see Figure 8(c) and 8(d). Needless to say Model B results in a higher budget than that actually spent, to yield a similar user profile as the historical one, except that in the recent years the number of users is 20% less.

Ratio = $\alpha_1/\alpha_2 = 100$ corresponds to 100 times higher importance over the number

\(^1\)Recall that users are measured in units of 1000 and budget in units of millions
of users. Essentially, this means that we are allowed to spend as much budget as we wish, but still within the imposed optimization criteria. In this case both models lead to a significantly higher budget. However, this is compensated for by much lower populations of users, for instance, 10,000 - 15,000 (Model A), and between 30,000 - 40,000 (Model B).

References


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**Appendix A**

**Necessary Conditions of Optimality for Problem (PA1)**

Direct substitution of the system equation

\[ x(t) = (a + be^{\alpha t}) e^{-\beta u(t)} \]  

(A.1)

into the cost in Problem (PA1) gives

\[ J(u) := \int_{t_0}^{t_f} \left[ \alpha_1 (a + be^{\alpha t}) e^{-\beta u(t)} + \alpha_2 u(t) \right] dt \]
The necessary condition of optimality is
\[ \frac{\partial L}{\partial u}(u) = 0 , \]
where \( L \) denotes the integrand above. This yields
\[ -\alpha_1 \beta (a + b e^{\alpha t}) e^{-\beta u(t)} + \alpha_2 = 0 . \tag{A.2} \]
After simplifying this for \( u \), the following control law is obtained.
\[ u(t) = \frac{1}{\beta} \ln \left( \frac{\alpha_1}{\alpha_2 \beta (a + b e^{\alpha t})} \right) . \tag{A.3} \]
Using A.1 and A.3 it is easy to see that optimal \( x \) is simply a constant depending on the ratio \( \frac{\alpha_1}{\alpha_2} \) and the efficiency of the budget \( \beta \), that is,
\[ x(t) = \frac{\alpha_2}{\alpha_1 \beta} . \]

**Necessary Conditions of Optimality for Problem (PB1)**

Given the system equation
\[ x(k) = \left( a + be^{\alpha k} \right) e^{-\beta u(k-1)} \tag{A.4} \]
we aim to minimize the cost function
\[ J(u) := \sum_{k=17}^{27} (\alpha_1 x(k) + \alpha_2 u(k)) \]
where \( x(17) = 49,000 \), the historical number of users in year 1987. The number of users in 1997, \( x(27) \), is free.

Hamiltonian for the problem is defined as
\[ H(x, u, \psi, k) := \alpha_1 x(k) + \alpha_2 u(k) + \psi(k+1) \left( a + be^{\alpha (k+1)} \right) e^{-\beta u(k)} \]
where, \( \psi(k) \) is the costate variable at time \( k \). The necessary conditions of optimality are given by the discrete-time maximum principle [20]:
\[ \frac{\partial H}{\partial u} = 0 \text{ and } \frac{\partial H}{\partial x} = \psi(k), \ k = 17, ..., 27. \]

The condition \( \partial H/\partial u = 0 \) results in
\[ \alpha_2 = \beta \psi(k+1) \left( a + be^{\alpha (k+1)} \right) e^{-\beta u(k)} \tag{A.5} \]
and the condition $\partial H/\partial x = \psi(k)$ yields
\[ \psi(k) = \alpha_1. \] (A.6)

Equations (A.5) and (A.6), yield the following dynamic control law
\[ u(k) = \frac{1}{\beta} \ln \left( \frac{\alpha_1}{\alpha_2} \beta \left( a + b e^{\alpha_1(k+1)} \right) \right). \] (A.7)

As before it can be seen that
\[ x(k) = \frac{\alpha_2}{\alpha_1 \beta}. \]

In fact, the optimal control in the linear cost case coincides with the direct control, while the constant-size user population is considered.

**Necessary Conditions of Optimality for Problem (PA2)**

Substitution of the system equation
\[ x(t) = (a + b e^{\alpha t}) e^{-\beta u(t)} \] (A.8)
into the cost in Problem (PA2) gives
\[ J(u) := \int_{t_0}^{T} \left[ \alpha_1 (a + b e^{\alpha t})^2 e^{-2\beta u(t)} + \alpha_2 u^2(t) \right] dt \]

The necessary condition of optimality yields
\[ -\alpha_1 \beta (a + b e^{\alpha t})^2 e^{-2\beta u(t)} + \alpha_2 u(t) = 0. \] (A.9)

Using (A.8) in (A.9) and rearranging one gets the feedback control law
\[ u(t) = \frac{\alpha_1}{\alpha_2} \beta x^2(t). \] (A.10)

A closed form expression for $u(t)$ can also be given using the so-called Lambert w function [7]. We write equation (A.9) as
\[ 2 \beta u(t) e^{2\beta u(t)} = 2 \frac{\alpha_1}{\alpha_2} \beta^2 \left( a + b e^{\alpha t} \right)^2 \]
so that
\[ u(t) = \frac{1}{2\beta} L_w \left[ 2\frac{\alpha_1}{\alpha_2} \beta^2 \left( a + b e^{\alpha t} \right)^2 \right] \] (A.11)
where $L_w$ denotes the Lambert w function.

**Necessary Conditions of Optimality for Problem (PB2)**
Given the system equation
\[ x(k) = \left( a + be^{\alpha k} \right) e^{-\beta u(k-1)} \] (A.12)
we wish to minimize the cost
\[ J(u) := \sum_{k=17}^{27} \left( \alpha_1 x^2(k) + \alpha_2 u^2(k) \right) \]
where \( x(17) = 49,000 \), the observed number of users in year 1987. The number of users in 1997, \( x(27) \), is free.

The Hamiltonian for the problem is
\[ H(x, u, \psi, k) := \alpha_1 x^2(k) + \alpha_2 u^2(k) + \psi(k+1) \left( a + be^{\alpha(k+1)} \right) e^{-\beta u(k)} \]
where, \( \psi(k) \) is the costate variable at time \( k \). By the discrete-time maximum principle [20], the necessary conditions of optimality are:
\[ \frac{\partial H}{\partial u} = 0 \text{ and } \frac{\partial H}{\partial x} = \psi(k), \quad k = 17, \ldots, 27. \]

The condition \( \partial H/\partial u = 0 \) results in
\[ 2 \alpha_2 u(k) e^{\beta u(k)} = \beta \psi(k+1) \left( a + be^{\alpha(k+1)} \right) \] (A.13)
and the condition \( \partial H/\partial x = \psi(k) \) yields
\[ \psi(k) = 2 \alpha_1 x(k). \] (A.14)

Using Equations (A.12), (A.13) and (A.14), one gets the dynamic feedback control law
\[ u(k) = \frac{\alpha_1}{\alpha_2} \beta x^2(k+1) . \] (A.15)

It is interesting to note that the optimal regime has the dynamics
\[ x(k+1) = \sqrt{\frac{\alpha_2}{\alpha_1 \beta}} u(k) . \]

A closed-form expression for \( u(k) \) can be obtained by means of the Lambert \( w \) function, after substituting (A.12) into (A.15) and rearranging, as
\[ u(k) = \frac{1}{2 \beta} L_w \left[ \frac{2 \alpha_1}{\alpha_2} \beta^2 \left( a + be^{\alpha(k+1)} \right)^2 \right]. \] (A.16)