The Leap-Frog Algorithm and Optimal Control: Theoretical Aspects

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Abstract

The Leap-Frog Algorithm was originally devised to find geodesics in connected complete Riemannian manifolds. A direct application of this algorithm to find optimal control for systems with unbounded control (without a convergence proof) gave promising results. This motivated us to go ahead with generalizing the mathematical rigour of the leap-frog algorithm to a class of optimal control problems.

In this work, we give a theoretical analysis of the Leap-Frog Algorithm for a class of optimal control problems with bounded controls in the plane. The Leap-Frog Algorithm assumes that the problem is already solved locally. This requirement translates to the case of optimal control as the availability of a local solution of the problem. This is related to the structure of the small-time reachable sets. We consider the local time-optimal bang-bang control solution of a system with bounded controls, which is well-understood in the plane.

Key words: Time-optimal control, Nonlinear systems.

1 Introduction and Background

The Leap-Frog Algorithm was originally devised to find geodesics in connected complete Riemannian manifolds (Noakes [5]). This algorithm gave promising results when it was applied to find an optimal control for a class of systems with unbounded control (Kaya and Noakes [2]). However no convergence proof was given for this application to optimal control.

In this work, a theoretical analysis of the Leap-Frog Algorithm is given for optimal control problems in $\mathbb{R}^2$ with bounded controls. The local time-optimal control solution of systems linear in the control input is well understood and is a bang-bang control with only one switching in the case where the objective is to go from one point to another in minimum time.

Consider the so-called linear analytic control system in the plane

$$x(t) = f(x(t)) + u(t)g(x(t))$$

with bounded single input $u = u(t)$ satisfying $|u| \leq 1$, where $x \in \mathbb{R}^2$, and where $f$ and $g$ are $C^1$ vector fields in $\mathbb{R}^2$. The control input function $u$ is an element from a space $\mathcal{U}$ of input functions. The space $\mathcal{U}$ is chosen to be compact by restricting the control to bang-bang with finite number of switchings. The domain of $u$ is some subset of $\mathbb{R}^{m+1}$, where each coordinate in $\mathbb{R}^{m+1}$ is the value of a switching time and $m$ is given as the...
maximum number of switchings. The magnitudes of the vector fields $f$ and $g$ are assumed to be bounded. In subsequent references to this system it will often be convenient to omit reference to the dependence on $t$.

1.1 Small-time reachable sets

Let $X$ be the vector field $f + u g$. The integral curve $\Gamma_{X,x_0}$ of $X$ starting at $x_0$ for positive times is the solution curve of the dynamical system (1.1) with initial condition $x(0) = x_0$. In other words, if $\Phi_{X,t} : V \to \mathbb{R}^n$, $V \subset \mathbb{R}^n$ is the flow of $X$ such that $x_0 \mapsto \Gamma_{X,x_0}(t)$, then $\Phi_{X,t}(x_0) = \Gamma_{X,x_0}(t)$, $\Gamma_{X,x_0}(0) = x_0$.

The following definitions are adopted from Krener and Schättler [4]. A trajectory of the system (1.1) corresponding to a control $u(\cdot)$ is a continuous curve $x(\cdot)$ satisfying $\dot{x}(t) = f(x(t)) + u(t) g(x(t))$ for almost all $t$. A point $x_1$ is reachable from a point $x_0$ within time $T$ if and only if, for some admissible control, there exists a trajectory $x(\cdot)$ defined on an interval $[0,T]$, such that $x(0) = x_0$ and $x(t) = x_1$. The set of all such points $x_1$ is denoted by $\mathcal{R}(x_0, \leq T)$. The reachable set $\mathcal{R}(x_0, \leq T)$ is said to be a small-time reachable set when $T$ is sufficiently small.

Let $X_+ = f + g$ and $X_- = f - g$ be the bang-bang vector fields, corresponding to the control input values $u = +1$ and $u = -1$, respectively. It is assumed that $X_+$ and $X_-$ are linearly independent at $x_0$. As pointed out in Krener and Schättler [4], the boundary of the small-time reachable set $\mathcal{R}(x_0, \leq T)$ consists of the integral curves $\Gamma_{X_+,x_0}$ and $\Gamma_{X_-,x_0}$, where $X_+$ and $X_-$ are linearly independent at $x_0$. So, $\mathcal{R}(x_0, \leq T)$ itself is the union of this boundary and the open sector between $\Gamma_{X_+,x_0}$ and $\Gamma_{X_-,x_0}$. A point $x_1$ in $\mathcal{R}(x_0, \leq T)$ can be reached by switching from $X_+$ to $X_-$ at some point $x_s$ along $\Gamma_{X_+,x_0}$. Small-time reachability is depicted in Figure 1. Note that one can also reach the same point by using first $X_-$ and then $X_+$.
1.2 Local optimal control

Consider the following control system.

\[ \dot{x} = v \, X_+(x) + (1 - v) \, X_-(x) \quad , \quad 0 \leq v \leq 1 \] (1.2)

where the new control input \( v \) is defined by \( v = (1 + u)/2 \). Note that System (1.2) is equivalent to System (1.1). By the Pontryagin Maximum Principle (PMP) (Pontryagin et al. [6]) it is well-known that the time-optimal control solution of System (1.1) or equivalently System (1.2) is bang-bang, if there exists no singularity. A singularity situation arises when the so-called switching function of the system vanishes for a finite interval of time. It is not an easy task in general to determine whether a given system has singular solutions or not. Even for the planar control systems, the necessary conditions for optimality given by the PMP may not yield such information.

Hájek [1] gives a theorem which states a set of necessary conditions of local optimality different from those given by the PMP. It concerns the following special class of optimal control problems.

\[ P : \begin{cases} \min_v \int_0^1 (v \phi(x) + (1 - v) \psi(x)) \, dt \\ \text{subject to} \quad \dot{x} = v \, X_+(x) + (1 - v) \, X_-(x) \quad , \quad 0 \leq v \leq 1 \\ \text{such that} \quad x(0) = x_0 , \quad x(1) = x_1 \end{cases} \]

where \( \phi, \psi : \mathbb{R}^2 \rightarrow \mathbb{R} \) are \( C^1 \) functions. As a special case, time-optimal control can also be considered by allowing the final time to be free and by setting \( \phi(x) = \psi(x) = 1 \). Then Problem (P) becomes

\[ P_t : \begin{cases} \min_v \int_0^T 1 \, dt \\ \text{subject to} \quad \dot{x} = v \, X_+(x) + (1 - v) \, X_-(x) \quad , \quad 0 \leq v \leq 1 \\ \text{such that} \quad x(0) = x_0 , \quad x(1) = x_1 \end{cases} \]

Unfortunately, the setting given in Hájek [1] does not accommodate the particularly interesting minimum energy problems with the cost functional \( \int_0^1 v^2 \, dt \) or \( \int_0^1 |v| \, dt \).

Let \( \Delta \) denote the determinant of the \( 2 \times 2 \) matrix \( [X_+ \quad X_-] \), namely

\[ \Delta(x) = \det [X_+ \quad X_-](x) \] (1.3)

where the vector fields \( X_+ \) and \( X_- \) are evaluated and expressed as column vectors \( X_+ = [(X_+)_1 \quad (X_+)_2]^T \) and \( X_- = [(X_-)_1 \quad (X_-)_2]^T \) at \( x \). Let \( \Omega \) denote the divergence

\[ \Omega(x) = \text{div} \left( \frac{\psi X_+ - \phi X_-}{\Delta(x)}(x) \right) \] (1.4)

In the case of Problem (\( P_t \)), \( \Omega \) becomes \( \Omega = \text{div} \left( (X_+ - X_-)/\Delta \right) \). Furthermore, it is not difficult to show that

\[ \Omega(x) = - \text{div} \left( \frac{g(x)}{\langle f, g^\perp \rangle(x)} \right) \] (1.5)

where \( g^\perp = [g_2 \quad - g_1] \), and \( \langle \cdot, \cdot \rangle \) is the Euclidean inner product. Note that \( \langle g, g^\perp \rangle = 0 \).
Theorem 1.1 (Necessary conditions for local optimality) (Hájek [1]) Consider the optimal control problem (P) or more specifically (P₁). If there exists a local optimum solution from $x₀$ to $x₁$, $x₁$ being sufficiently close to $x₀$, then the control $v$ is bang-bang with one switching, i.e. $v$ takes either the sequence of values \{0, 1\} or \{1, 0\}, and the conditions

$$\Delta(x₀) \neq 0 \quad \text{(1.6)}$$

and

$$\Omega(x₀) \neq 0 \quad \text{(1.7)}$$

hold.

Condition (1.6) is equivalent to the linear independence of $X_+$ and $X_-$, or of $f$ and $g$, at $x₀$. Recall that this condition is necessary for the small-time reachable set $\mathcal{K}(x₀, ≤ T)$ to be well-defined. Both conditions (1.6) and (1.7) along with $v$ being bang-bang with one switching are necessary for the existence of a local optimal control. It should be noted that in both Problems (P) and (P₁) the terminal point $x₁$ is chosen so close to $x₀$ that the optimal control is guaranteed to be local, or in other words, Theorem 1.1 holds. There is no doubt that $x₁$ has to be chosen in the reachable set; however, it requires further scrutiny to find out the meaning of the condition (1.7) within the context of reachable sets. Such a discussion is beyond the scope of the present paper.

1.3 The Leap-Frog Algorithm

The Leap-Frog Algorithm has been described in detail in a complementary paper, “The Leap-Frog Algorithm and optimal control: background and demonstration,” (Kaya & Noakes [3]). The details of the algorithm will not be repeated here.

In this paper we follow the setting and methodology used by Noakes [5] in his discussion of geodesics in connected and complete Riemannian manifolds.

2 Definitions and Tools

Let $\mu$ be a piecewise-$C^1$ trajectory such that $\mu(0) = x₀$ and $\mu(1) = x₁$. The final point $x₁$ is not necessarily close to $x₀$. The trajectory $\mu$ is not necessarily optimal either; it is simply a feasible curve between $x₀$ and $x₁$. It is parameterized to the time required to travel from $x₀$ to $x₁$ along $\mu$ itself. The aim is to find a time-optimal trajectory between ‘distant’ points $x₀$ and $x₁$.

We initially partition the feasible (but not optimal) curve $\mu$ so as to obtain $q$ pieces of the trajectory. Choose $0 = t₀ < t₁ < \ldots < t_q = 1$ so that the partition points $\mu(t_{i-1})$ and $\mu(tᵢ)$, $i = 1, \ldots, q$, are close enough to each other and that there exists a local optimal solution between $\mu(t_{i-1})$ and $\mu(tᵢ)$, $i = 1, \ldots, q-1$, which is easily computable. Concerning the local time-optimal control, we assume that the linear independence and divergence conditions given in Theorem 1.1 are satisfied at each partition point. Then the optimal control is bang-bang with one switching from $\mu(t_{i-1})$ to $\mu(tᵢ)$. So there are two possibilities for the choice of the sequence of vector fields: either $\{X_+, X_-\}$ with $u = +1, -1$, or $\{X_-, X_+\}$ with $u = -1, +1$.

Let $\tau : \mathbb{R}^2 × \mathbb{R}^2 \rightarrow \mathbb{R}$ denote the minimum-time function such that $\tau(\mu(t_{i-1}), \mu(tᵢ))$ is the minimum time to get from $\mu(t_{i-1})$ to $\mu(tᵢ)$. Recall that $\mu(tᵢ)$ is reachable from $\mu(t_{i-1})$ through bang-bang control with one switching. So the function $\tau$ is continuous. Let
\[ \gamma : [0,1] \to \mathbb{R}^2 \] be a **minimum-time trajectory** so that \( \gamma(0) = \mu(t_{i-1}) \) and \( \gamma(1) = \mu(t_i) \). Note that \( \gamma \) is parameterized to \( \tau(\mu(t_{i-1}), \mu(t_i)) \). Let \( D \) denote the set

\[
D = \{(y, z) \in \mathbb{R}^2 \times \mathbb{R}^2 : z \in \mathcal{R}(y, \leq 2\delta)\}
\]

where \( \delta > 0 \) is so chosen that there exists a local time-optimal control from \( y \) to \( z \). Note that if \( (y, z) \in D \) then \( \tau(y, z) \leq 2\delta \). Define a *midpoint map* to be any \( M : D \to \mathbb{R}^2 \) such that \( M(y, z) = \gamma(1/2) \) for some local time-optimal trajectory \( \gamma : [0,1] \to \mathbb{R}^2 \). The midpoint map \( M \) is not continuous.

Let \( X \subset \mathbb{R}^2 \). Let \( X_- \) be the set of all \( q + 1 \)-tuples \( y = (y_0, y_1, \ldots, y_q) \in X^{q+1} \) where

\[
\tau(y_{i-1}, y_i) \leq \delta \quad \text{for all} \quad i = 1, \ldots, q - 1.
\]

For \( 1 < p < q \) define \( G_p : Y \to X^{q+1} \) by

\[
G_p(y) = (y_0, \ldots, y_{p-1}, z_p, y_{p+1}, \ldots, y_q)
\]

where \( z_p = M(y_{p-1}, y_{p+1}) \).

**Lemma 2.1** \( G_p(y) \in Y \).

**Proof:**

\[
\tau(y_{p-1}, z_p) = \tau(z_p, y_{p+1}) = \frac{\tau(y_{p-1}, y_{p+1})}{2} \leq \frac{\tau(y_{p-1}, y_p) + \tau(y_p, y_{p+1})}{2} \leq \frac{2\delta}{2}.
\]

\[ \Box \]

So \( G_p : Y \to Y \) where \( 1 < p < q \). Since \( M \) is not continuous, neither are the \( G_p \).

Define \( F : Y \to Y \) as the composite

\[
F = G_{p-1} \circ G_{p-2} \circ \cdots \circ G_1.
\]

Note that \( F \) is also not continuous. The \( q + 1 \)-tuple \( z = F(y) \in Y \) can be defined alternatively by

(i) \( z_0 = y_0 \)

(ii) \( z_i = M(z_{i-1}, y_{i+1}) \) for \( 1 \leq i \leq q \)

(iii) \( z_q = y_q \).

It is interesting to note that \( F(y) \) does not depend on \( y_1 \).

The \( q + 1 \)-tuple \( y \) is said to have the **time-length** \( \alpha \) defined by

\[
\alpha(y) = \sum_{i=1}^{q} \tau(y_{i-1}, y_i)
\]

and hence

\[
\tau(y_0, y_q) \leq \alpha(y).
\]

Note that the time-length function \( \alpha : Y \to \mathbb{R} \) is continuous, because \( \tau \) is continuous.
Lemma 2.2 For $1 \leq p \leq q$ and all $y \in Y$
\[ \alpha(G_p(y)) \leq \alpha(y) . \]

Proof:
\[ \alpha(y) - \alpha(G_p(y)) = \tau(y_{p-1}, y_p) + \tau(y_p, y_{p+1}) - \tau(y_{p-1}, y_{p+1}) \geq 0 . \]

If one considers the composite $F = G_{p-1} \circ G_{p-2} \circ \cdots \circ G_1$ in Lemma 2.2 instead of $G_p$, then the following lemma is obtained

Lemma 2.3 For all $y \in Y$
\[ \alpha(F(y)) \leq \alpha(y) . \]

3 A Convergence Analysis

Let $y \in Y$ and define $s^{(n)} = F^n(y)$ for $n \geq 1$. By Lemma 2.3 the sequence $\{\alpha(s^{(n)}) : n \geq 1\}$ converges to its infimum $\alpha^{(\infty)}$ with
\[ \alpha^{(\infty)} \in [\tau(y_0, y_q), q\delta] . \]  \hspace{1cm} (3.1)

Let $Y_{T_1}$ be the set $Y$ for which $\delta = T_1/q$, where $T_1$ is the total time travelled along the initial feasible trajectory, and $q$ the number of partitions. Then $F$ maps $Y_{T_1}$ to itself.

Lemma 3.1 $Y_{T_1}$ is compact.

Proof: Since the right-hand side of the control system (1.1) is bounded in magnitude, the set $Y_{T_1}$ is also bounded. Now one needs to show that $Y_{T_1}$ is closed as well. Consider $y = (y_0, y_1, \ldots, y_q) \in Y_{T_1}$. Note that $\hat{y}_i^{(n)} \in \mathcal{R}(\hat{y}_{i-1}^{(n)}, \leq \delta)$ for the $n$-th iterate $y^{(n)} = F^{(n)}(y)$; in other words,
\[ y_i^{(n)} = y_{i-1}^{(n)} + \int_0^{\tau^{(n)}} \left( f(x(t)) + u^{(n)}(t)g(x(t)) \right) dt \]
where $\tau^{(n)} = \tau(y_{i-1}^{(n)}, y_i^{(n)})$. Since the input function space $\mathcal{U}$ was chosen to be compact, one can take a convergent subsequence of $u^{(n)}(t)$, and take the limit as $n \rightarrow \infty$ to get
\[ y_i^{(\infty)} = y_{i-1}^{(\infty)} + \int_0^{\tau^{(\infty)}} \left( f(x(t)) + u^{(\infty)}(t)g(x(t)) \right) dt . \]
So $y_i^{(\infty)} \in \mathcal{R}(y_{i-1}^{(\infty)}, \leq \delta)$, and thus $Y_{T_1}$ is closed. \hspace{1cm} \square

Now, since $Y_{T_1}$ is compact, there exists a convergent subsequence $\{s^{(n_j)} : j \geq 1\}$ of the sequence $\{s^{(n)} : n \geq 1\} \subset Y_{T_1}$. Write
\[ \lim_{j \rightarrow \infty} s^{(n_j)} = s^{(\infty)} \in Y_{T_1} \]
where
\[ s^{(\infty)} = \left( s_0^{(\infty)}, s_1^{(\infty)}, \ldots, s_q^{(\infty)} \right). \]

Now, since \( \alpha \) is continuous
\[ \alpha(s^{(\infty)}) = \alpha^{(\infty)}. \]

The curve \( \gamma : [0, 1] \rightarrow \mathbb{R}^2 \) is a minimum-time trajectory from a given point \( x \in \mathbb{R}^2 \) to another, \( y \in \mathbb{R}^2 \), such that \( \gamma(0) = x \) and \( \gamma(1) = y \), \( \gamma \) being parameterized to the time traveled from \( x \) to \( y \). The point \( z \) is said to be between \( x, y \), if \( z \) lies in the image of \( \gamma \).

Note that, trivially, \( x \) and \( y \) are between \( x, y \).

The \( q \)-tuple \( w \in Y_{\gamma_1} \) is said to be extreme when \( w_i \) is between \( w_{i-1}, w_{i+1} \) for all \( 1 < i < p \).

Let \( \mu_i : [0, 1] \rightarrow \mathbb{R}^2 \) be the minimum-time trajectory from \( w_{i-1} \) to \( w_i \), \( i = 1, \ldots, q \). Then \( \mu_w : [0, 1] \rightarrow \mathbb{R}^2 \) is defined as the concatenation of \( \mu_1, \mu_2, \ldots, \mu_q \) in the given order such that \( \mu_w(0) = \mu_1(0), \mu_w(i/q) = \mu_i(1) = \mu_{i+1}(0), i = 1, \ldots, q - 1 \), and \( \mu_w(1) = \mu_q(1) \).

A critical trajectory of the control system (1.1) from one point to another is a trajectory which is feasible and satisfies the PMP.

In this work it is assumed that none of the points \( w_1, w_2, \ldots, w_q \) coincide with each other. In the case when two or more of \( w_i \) become the same point, which is referred to as a multiplicity in Noakes [3], the analysis is very involved. Noakes [5] incorporates multiplicities in the theory of the algorithm he gives for the geodesics. In this paper, it is assumed that each point has multiplicity one.

Lemma 3.2 The \( q \)-tuple \( w \in Y_{\gamma_1} \) is extreme if and only if \( \mu_w \) is a critical trajectory.

Proof: Let \( \gamma_i : [0, 1] \rightarrow \mathbb{R}^2 \) denote a minimum-time trajectory from \( w_{i-1} \) to \( w_{i+1} \) parameterized to \( \tau(w_{i-1}, w_{i+1}), i = 1, \ldots, q - 1 \). Since \( w_{i-1} \) and \( w_{i+1} \) are close enough to each other, a local time-optimal control exists from \( w_{i-1} \) to \( w_{i+1} \), i.e. bang-bang control with one switching. This type of local control is unique except in one rather rare situation: when the control sequences \( \{+1, -1\} \) and \( \{-1, +1\} \) result in the same total time. Without loss, in such a case, one may assume the control sequence to be \( \{+1, -1\} \). Then the local time-optimal control becomes unique.

Suppose that \( \mu_w \) is a critical trajectory. Then the trajectory from any \( w_{i-1} \) to \( w_{i+1} \), \( i = 1, \ldots, q \), is minimum-time (or equivalently the local minimum-time) which implies that \( w_i \) is between \( w_{i-1}, w_{i+1} \). Therefore \( w \) is extreme. This completes one part of the proof.

Suppose that \( w \) is extreme. Then we prove by induction that \( \mu_w \) is critical.

Let \( q = 2 \). Then \( \mu_w = \gamma_1 \), which is a minimum-time trajectory, and thus critical.

Assume that the lemma holds for \( q \leq k \). (Note that \( \mu_w \) is in this case composed by concatenating \( \mu_i, i = 1, \ldots, k \)). Denote \( \mu_w \) in this case by \( \mu_w^k \).

Let \( q = k + 1 \). Note that \( \gamma_k \) from \( w_{k-1} \) to \( w_{k+1} \) contains \( w_k \) because \( w \) is extreme, and \( \gamma_k \) is a minimum-time trajectory. Note that \( \mu_w^k \) is critical, i.e. it satisfies the PMP. Portions of the curves \( \mu^k \) and \( \mu^k_w \) between \( w_{k-1} \) and \( w_k \) overlap, so both curves satisfy the system equation (1.1) with the same initial conditions for that portion of the curve and thus satisfy the PMP. Therefore \( \mu_w^{k+1}, \) which is the concatenation of \( \mu_i, i = 1, \ldots, k + 1 \) satisfies the PMP and is a critical trajectory. This completes the proof. \( \square \)

The first two items of the following result come from Lemma 2.3.
Lemma 3.3  
(i) $\alpha(F(y)) \leq \alpha(y)$ .

(ii) if $\mu_y$ is a critical trajectory then $\alpha(F(y)) = \alpha(y)$ .

(iii) if $\alpha(F(y)) = \alpha(y)$ then $\mu_y$ is a critical trajectory.

Theorem 3.1 $\mu_{s(\infty)}$ is a critical trajectory.

Proof: Note that $F(s(\infty)) = s(\infty)$. Then

$$\alpha(F(s(\infty))) = \alpha(s(\infty))$$

and the theorem follows from Lemma 3.3.

References


